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Composite Failure Criterion - Probabilistic Formulation and Geometric Interpretation

by

Lim, Jong Chun

December 1988

Thesis Advisor: Edward M. Wu

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In applications, given the experimental measurements of the necessary statistical parameters for the specific composite, the probabilistic criterion of the composite failure and the reliability of the specific structure can be predicted.

Graphical representations for the joint reliability and joint failure contours were made in two and three dimensional space for the several different sets of statistical strength parameters to illustrate the effect of parameters on reliability.

Such understanding will enhance selection of fiber and matrix which have their own statistical strength parameters and can lead to improvement in reliability of some composite components in an aircraft structure.

These reliability and failure concepts can also be used in repair problems.

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Composite Failure Criterion - Probabilistic Formulation and Geometric Interpretation

by

Lim, Jong Chun
Major, Republic Of Korea Air Force
B. S., Korea Air Force Academy, Seoul, 1980

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Author : ______________________
Lim, Jong Chun

Approved by : ______________________
Edward M. Wu, Thesis Advisor

____
E. Roberts Wood, Chairman,
Department of Aeronautics

____
Gordon E. Schacher
Dean of Science and Engineering

ABSTRACT

The objective of this investigation is to derive the reliability and the associated probabilistic failure criterion for composite materials under combined stress. In the analytical derivation, the concept of joint probability was used and applied to the Weibull distribution function. In applications, given the experimental measurements of the necessary statistical parameters for the specific composite, the probabilistic criterion of the composite failure and the reliability of the specific structure can be predicted.

Graphical representations for the joint reliability and joint failure contours were made in two and three dimensional space for the several different sets of statistical strength parameters to illustrate the effect of parameters on reliability. Such understanding will enhance selection of fiber and matrix (which have their own statistical strength parameters) and can lead to improvement in reliability of some composite components in an aircraft structure. These reliability and failure concepts can also be used in repair problems by selecting the proper composite material with the appropriate statistical parameters.

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TABLE OF CONTENTS

I. INTRODUCTION

A. FAILURE AND RELIABILITY OF THE COMPOSITE MATERIAL

In a composite, the matrix and reinforcement materials are combined to form a macroscopically multiphase material . The macroscopic state gives rise to a material which is internally redundant against local defects and damages. [Ref.1] The ply is the basic unit in constructing composites which consists of fibers embedded in a matrix. It is possible, therefore, to relate composite design properties to corresponding constituent properties. As a result, it is possible to design structural components made of fiber-composite when the design requirements and the properties of the candidate constituent materials are known. For high performance structures, the unidirectional fiber-matrix composites consist of strong stiff fibers embedded in a comparatively ductile matrix . For a rational structural design, the strength of such unidirectional ply needs to be expressed in terms of the stress tensor. Because the ply is composed of numerous fibers, the composite failure process is sequential, therefore ad hoc failure criteria can not be applied to analyze the composite failure. Furthermore, such conventional failure criteria such as Mises and Tresca's criteria are related only to a single parameter (mean function of failure). As a result, they can not be applied to multiple parameter functions, so suitable general relations to represent failure contours of composite materials are required.

In the application of a one-dimensional failure criterion, the magnitude of stress (one component) at each spatial location of the structure is mapped into respective points on the stress space (which is a line). If all the points within the domain are bounded by the tensile strength limit and the compressive strength limit, then the entire structure may be considered safe. The reliability under unidirectional stress, then, is the product of the reliability of each spatial segment at the respective stress level. Under planar combined stress, the magnitude of stresses (3 components) at each spatial location of the structure is mapped into respective points on the six dimensional stress hyperspace. So the stress that is interior to a domain bounded by the failure surface is safe and the failure criterion is the analytical, graphical, or numerical representation of the failure surface. To formulate the failure criterion, consistency of mathematical operations can be assured by adhering to the quotient rule of tensor variables. Historically, the formulation of many failure criteria were intuitively taken to be based on failure mechanisms but in fact, they are phenomenological because of the inconsistency between macro and microscopic failure modes. Phenomenological failure criterion may be considered as a mathematical model of the material transfer function relating the external excitation (stresses) to the material's response (failure). The phenomenological mathematical model is intended to aid experimental design to facilitate interpolation, correlation, and retrieval of experimental observations. They do not, in general, address the statistical variability of strength.

For a combined stress failure mechanism, when one stress component does not affect the strength of another component, the failure is mechanistically uncoupled; conversely when one stress component affects the strength of another component, the failure is mechanistically coupled. In addition, if probabilistic failure is also taken into consideration, the condition for the failure mechanism is not fixed; it depends on the stochastic combinations of the intrinsic strengths. Under combined stress, mechanistic uncoupling with statistic independence can interact to produce phenomenologically coupled contours. On the other hand, mechanistically coupling with statistical dependency can also produce phenomenologically coupled statistical contours. These cases can only be distinguished by examination of both the contour in the stress space and the statistical strength distribution in the probability space.

In any phenomenological failure criterion, the shape of the failure envelope is *never completely* known until experiments are performed for all possible combined states of stress. As a result, complete experimental determination in two or more dimensions place an impractical demand on time and resources because of the large number of tests needed. However, with the establishment of a probabilistic failure criterion, this experimental failure contour can be related to the shape parameters (the variability) in the material's principal direction. With such supplemental information, failure contours connecting the mean value can be established by a smaller number of samples. Therefore an analytical model can complement the failure characterization by reducing the required number of experimental measurements, providing bases for comparison of the material's performance, and simplifying data and data base reduction. In structural analysis applications, the reliability of a specific structure under a combined stress can be determined by mapping the spatial domain into the stress domain through stress analysis. Using the probability failure criterion, the reliability of each spatial location corresponding to the respective stress tensor and magnitude can be calculated. The joint reliability of the individual spatial location can in turn be combined to predict the reliability of the entire structure.

B. OBJECTIVE OVERVIEW

The objective of this investigation is to derive an analytical model for composites failure and reliability under combined stress conditions. The Weibull distribution function was used in the probabilistic modeling. Chapter 2 presents the theoretical formulation for failure and reliability, and the resulting equations are shown graphically in the following chapters. Chapter 3 shows the joint reliability and linearized failure Cumulative Distribution Function (CDF) in two and three dimensional space. The effects of parameters which depend on the material were analyzed. Chapter 4 shows the joint failure CDF in two and three dimensional space. The effects of material parameters were also analyzed. In Appendix B, these probability distributions were compared to experimental data availabe in the literature.

Finally, Chapter 5 provides conclusions and some remarks for the future work in this area.

II. THEORETICAL FORMULATION

A. RANDOM VARIABLES

A failure criterion for a solid prescribes the risk of failure in terms of the causes of the failure. Mechanical catastrophic rupture is considered herein. The effect of mechanical failure may be considered to be caused by either stress or stain depending on the microscopic failure processes. If the constitutive relation of the material is available, conversion between stress and strain can be readily made and the two variables may be considered as equivalent. Similarly, a probabilistic failure criterion which describes the probable risk of failure can be expressed in terms of stress or strain as random variables. Stress is selected as the random variable herein with the understanding that the variable can be changed to strain with the appropriate constitutive relation. We denote the probabilistic random variable as X_i, for the scalar component of the stress tensor , and x_i for the realized random variable (i.e. the strength X_i under the respective stress component σ_i).

B. PROBABILISTIC CONDITIONS OF FAILURE

Under combined stress conditions, when more than one component of the stress tensor assumes non-zero values, the effect of each component of the stress tensor contributes to the failure of the composite. Failure of the composite does not occur if failure by each individual stress component does not occur.

For the composite subjected to a planar state of stress, we denote the failure events as:

A: Failure caused by normal stress along the fiber direction - σ_1

B: Failure caused by normal stress perpendicular to the fiber direction - σ_2

C: Failure caused by shear stress along the fiber direction - σ_6.

Then the failure of the composites is either by A or by B or by C or by all three as illustrated in Figure 2-1:

$$P(A + B + C) = P(A) + P(B) + P(C) - \{P(AB) + P(BC) + P(CA)\} + P(ABC) \tag{2-1}$$

As a special case, if all three events are mutually exclusive, then Equation (2-1) can be simplified by

$$P(A + B + C) = P(A) + P(B) + P(C) \tag{2-2}$$

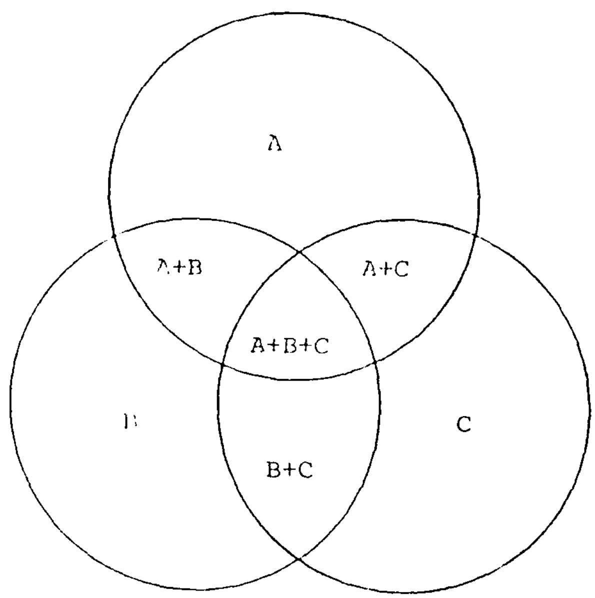

Figure 2-1 : Joint Probability

C. JOINT DISTRIBUTION FOR FAILURE

1. Discrete Case

In the development of the analytical expressions for the probability of failure, we first developed the general form of the distribution for three random variables using the notation X_i for the random variables and x_i *for the realized* random variables. For the planar stress, the customary notation for the range of $i = 1,2,6$ is used. These general expressions are then specialized for the combined stress cases by replacing the realized random variables by σ_i.

The probability of failure under the combined influence of three random variables can be expressed by the joint probability density function (pdf) of the individual realized random variables:

$$P(X_1 = \sigma_1, X_2 = \sigma_2, X_6 = \sigma_6) = f(\sigma_1, \sigma_2, \sigma_6) \tag{2-3}$$

where $f(\sigma_1, \sigma_2, \sigma_6) \geq 0$ and $\sum\sum\sum f(\sigma_1, \sigma_2, \sigma_6) = 1$. If random variable X_6 is to take on any one of the values of $\sigma_{61}, \sigma_{62}, \sigma_{63}, \ldots, \sigma_{6n}$, then the joint pdf at some combination of $X_1=\sigma_{1i}$, $X_2=\sigma_{2j}$, and $X_6=\sigma_{6k}$ can be expanded by

$$P(X_1 = \sigma_{1i}, X_2 = \sigma_{2j}, X_6 = \sigma_{6k}) = f(\sigma_{1i}, \sigma_{2j}, \sigma_{6k}) \tag{2-4}$$

$$P(X_1 = \sigma_{1i}, X_2 = \sigma_{2j}) = f(\sigma_{1i}, \sigma_{2j}) = \sum_{k=0}^{n} f(\sigma_{1i}, \sigma_{2j}, \sigma_{6k}) \tag{2-5}$$

$$P(X_2 = \sigma_{2j}, X_6 = \sigma_{6k}) = f(\sigma_{2j}, \sigma_{6k}) = \sum_{i=0}^{l} f(\sigma_{1i}, \sigma_{2j}, \sigma_{6k}) \tag{2-6}$$

$$P(X_1 = \sigma_{1i}, X_6 = \sigma_{6k}) = f(\sigma_{1i}, \sigma_{6k}) = \sum_{j=0}^{m} f(\sigma_{1i}, \sigma_{2j}, \sigma_{6k}) \tag{2-7}$$

and the marginal probability function can be obtained for the three random variable case:

$$P(X_1 = \sigma_{1i}) = f(\sigma_{1i}) = \sum_{j=0}^{m}\sum_{k=0}^{n} f(\sigma_{1i}, \sigma_{2j}, \sigma_{6k}) \tag{2-8}$$

$$P(X_2 = \sigma_{2j}) = f(\sigma_{2j}) = \sum_{i=0}^{l}\sum_{k=0}^{n} f(\sigma_{1i}, \sigma_{2j}, \sigma_{6k}) \tag{2-9}$$

$$P(X_6 = \sigma_{6k}) = f(\sigma_{6k}) = \sum_{i=0}^{l}\sum_{j=0}^{m} f(\sigma_{1i}, \sigma_{2j}, \sigma_{6k}) \tag{2-10}$$

which can be written

$$\sum_{i=0}^{l} f_1(\sigma_{1i}) = 1, \sum_{j=0}^{m} f_2(\sigma_{2j}) = 1, \sum_{k=0}^{n} f_6(\sigma_{6k}) = 1 \tag{2-11}$$

$$\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} f(\sigma_{1i}, \sigma_{2j}, \sigma_{6k}) = 1 \tag{2-12}$$

2. Continuous Case

If three random variables are considered, then Equation (2-4) through (2-7) can be expanded by

$$P(X_1 \le \sigma_1, X_2 \le \sigma_2, X_6 \le \sigma_6) = F(\sigma_1, \sigma_2, \sigma_6)$$

$$= \int_{-\infty}^{\sigma_6}\int_{-\infty}^{\sigma_2}\int_{-\infty}^{\sigma_1} f(u, v, w)\,du\,dv\,dw \tag{2-13}$$

$$P(X_1 \le \sigma_1, X_2 \le \sigma_2) = F(\sigma_1, \sigma_2) = \int_{-\infty}^{\sigma_2}\int_{-\infty}^{\sigma_1}\int_{-\infty}^{\infty} f(u, v, w)\,dw\,du\,dv \tag{2-14}$$

$$P(X_2 \le \sigma_2, X_6 \le \sigma_6) = F(\sigma_2, \sigma_6) = \int_{-\infty}^{\sigma_6}\int_{-\infty}^{\sigma_2}\int_{-\infty}^{\infty} f(u, v, w)\,du\,dv\,dw \tag{2-15}$$

$$P(X_1 \le \sigma_1, X_6 \le \sigma_6) = F(\sigma_1, \sigma_6) = \int_{-\infty}^{\sigma_6}\int_{-\infty}^{\sigma_1}\int_{-\infty}^{\infty} f(u, v, w)\,dv\,du\,dw \tag{2-16}$$

and the marginal distribution can be obtained by

$$P(X_1 \le \sigma_1) = F(\sigma_1) = \int_{-\infty}^{\sigma_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u, v, w)\,dv\,dw\,du \tag{2-17}$$

$$P(X_2 \leq \sigma_2) = F(\sigma_2) = \int_{-\infty}^{\sigma_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v,w)\,dw\,du\,dv \tag{2-18}$$

$$P(X_6 \leq \sigma_6) = F(\sigma_6) = \int_{-\infty}^{\sigma_6}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v,w)\,du\,dv\,dw \tag{2-19}$$

which can be written

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v,w)\,du\,dv\,dw = 1 \tag{2-20}$$

From the definition of conditional probability, the conditional probability of σ_2 given that σ_1 occurs can be represented by a function form:

$$f(\sigma_2|\sigma_1) = \frac{f(\sigma_1,\sigma_2)}{f_1(\sigma_1)} \tag{2-21}$$

where $f(\sigma_1, \sigma_2) = P(X_1=\sigma_1, X_2=\sigma_2)$

a. General Case

If we consider the combined stress case, the random variables X_1, X_2, and X_6 can be assumed to be the failure strength in the 1, 2, and 6 direction respectively. And under a combined stress condition, the failure occurs when either failure strength X_1, X_2, or X_6 is reached. Therefore the probability of failure under combined stress X_1 and X_2 can be represented by

$$P(X_1 + X_2) = P(X_1) + P(X_2) - P(X_1, X_2)$$

and then the above relation can be related to the joint distribution function as follows

$$F(\sigma_1, \sigma_2) = P(X_1 \leq \sigma_1, X_2 \leq \sigma_2) = P(X_1 + X_2;\ X_1 \leq \sigma_1, X_2 \leq \sigma_2) \tag{2-22}$$

and similarly for three random variables case.

$$F(\sigma_1, \sigma_2, \sigma_6) = P(X_1 \leq \sigma_1, X_2 \leq \sigma_2, X_6 \leq \sigma_6) = P(X_1 + X_2 + X_6 : X_1 \leq \sigma_1, X_2 \leq \sigma_2, X_6 \leq \sigma_6) \tag{2-23}$$

Equation (2-22) is proven in Appendix A-1 using the Weibull distribution function for the independent case.

If the joint distribution of failure under the combined stresses of σ_1, σ_2, and σ_6 is considered, the strength failure criterion of a composite is the minimum of the failure strength X_1, X_2, and X_6, that is, at $X_c = X_1 \wedge X_2 \wedge X_6$. (We are not dealing with the minimum magnitude among X_1, X_2, and X_6 but rather one value among X_1, X_2, and X_6 which is less than the intrinsic failure strengthes of σ_1, σ_2, and σ_6 becomes the minimum value here.) So

$$F(\sigma_1, \sigma_2, \sigma_6) = P\{X_c = X_1 \wedge X_2 \wedge X_6 \text{over} (\sigma_1, \sigma_2, \sigma_6)\}$$

where (σ_1, σ_2, σ_6) is the three dimensional stress domain. If the relations among σ_1, σ_2 and σ_6 are given by

$$\sigma_1 = B_{12}\sigma_2, \ \sigma_2 = B_{26}\sigma_6, \ \sigma_6 = B_{61}\sigma_1 \tag{2-24}$$

then, the joint pdf $P(X_1 \leq \sigma_1, X_2 \leq \sigma_2, X_6 \leq \sigma_6)$ can be evaluated. If σ_6 is assumed to be zero, then the joint probability functions, $P(X_1 \leq \sigma_1, X_2 \leq \sigma_2)$ can be divided into two probability distribution functions corresponding to S_1 and S_2 domains as shown in Figure 2-2. The domain S_1 is applied for the case when $[X_1 / X_2 \leq B_{12}$ and $X_1 \leq \sigma_1]$ which the composite fails by failure strenth $X_2 = X_1 \wedge X_2$, whereas the domain S_2 is applied for the case when $[X_1/X_2 \geq B_{12}$ and $X_2 \leq \sigma_2]$ which the composite fails by failure strength $X_1 = X_1 \wedge X_2$. So the following joint relation can be obtained:

$$\begin{aligned} P(X_1 \leq \sigma_1, X_2 \leq \sigma_2) &= P(X_1/X_2 \leq B_{12} \text{ and } X_1 \leq \sigma_1) \\ &+ P(X_1/X_2 \geq B_{12} \text{ and } X_2 \leq \sigma_2) \end{aligned} \tag{2-25}$$

Applying Equation (2-25) to domains described in Figure 2-2, then

$$P(X_1 \leq \sigma_1, X_2 \leq \sigma_2) = \int_{S_1}\int f(u,v)dudv + \int_{S_2}\int f(u,v)dudv$$

Using end limits and considering $P(X_1 \le \sigma_1, X_2 \le \sigma_2) = F(\sigma_1, \sigma_2)$, then the joint distribution function for the general case can be represented by

$$F(\sigma_1, \sigma_2) = \int_0^{\sigma_1} \int_{\frac{u}{B_{12}}}^{\infty} f(u,v)\,dv\,du + \int_0^{\sigma_2} \int_{B_{12}v}^{\infty} f(u,v)\,du\,dv \qquad (2\text{-}26)$$

where u and v are dummy variables.

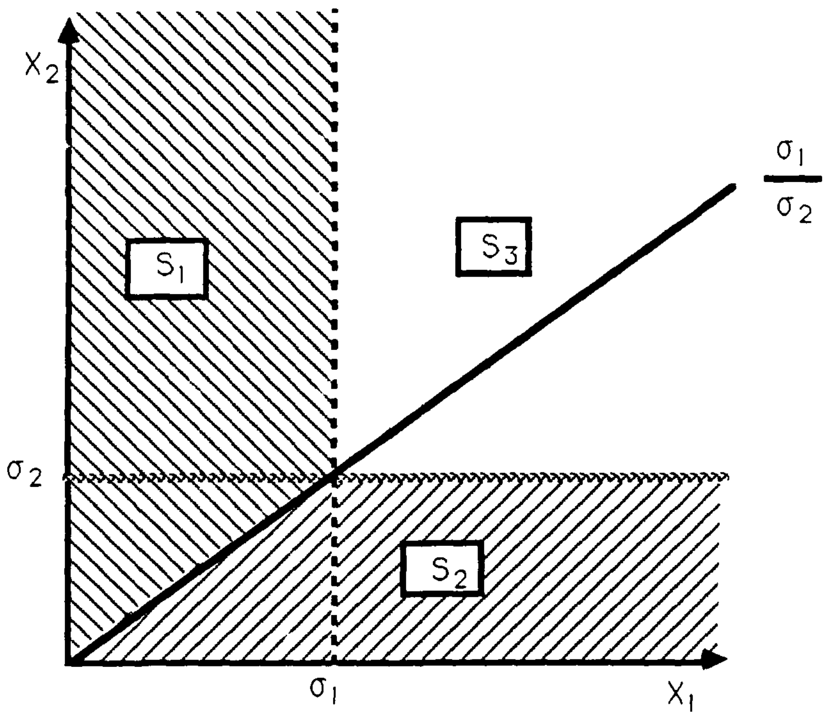

Figure 2-2 : Domains For The Joint Distribution

b. Specific Case

The joint probability of the bi-axial case (Equation (2-25)) can be expanded for specific cases.

(1) Independent Case. If the events $X_1 = \sigma_1$ and $X_2 = \sigma_2$ are independent for all σ_1 and σ_2, then the joint probability can be represented by

$$P(X_1=\sigma_1, X_2=\sigma_2) = P(X_1=\sigma_1)\, P(X_2=\sigma_2)$$

or equivalently

$$f(\sigma_1,\sigma_2) = f_1(\sigma_1)\, f_2(\sigma_2) \tag{2-27}$$

If three independent random variables are considered:

$$P(X_1=\sigma_1, X_2=\sigma_2, X_6=\sigma_6) = P(X_1=\sigma_1)\, P(X_2=\sigma_2)\, P(X_6=\sigma_6)$$

$$f(\sigma_1,\sigma_2,\sigma_6) = f(\sigma_1)\, f(\sigma_2)\, f(\sigma_6) \tag{2-28}$$

Similarly, if the random variable X_1 and X_2 are independent for all σ_1 and σ_2:

$$P(X_1 \leq \sigma_1, X_2 \leq \sigma_2) = P(X_1 \leq \sigma_1)\, P(X_2 \leq \sigma_2)$$

or equivalently:

$$F(\sigma_1,\sigma_2) = F_1(\sigma_1)\, F_2(\sigma_2) \tag{2-29}$$

where $F_1(\sigma_1)$ and $F_2(\sigma_2)$ are the marginal distribution functions of X_1 and X_2 respectively. Similarly, for the three independent random variable case:

$$F(X_1 \leq \sigma_1, X_2 \leq \sigma_2, X_6 \leq \sigma_6) = P(X_1 \leq \sigma_1)\, P(X_2 \leq \sigma_2)\, P(X_6 \leq \sigma_6)$$

or equivalently

$$F(\sigma_1,\sigma_2,\sigma_6) = F(\sigma_1)\, F(\sigma_2)\, F(\sigma_6) \tag{2-30}$$

So the joint distribution function can be expanded a step further for the independent case.

From Equation (2-26) (refer to Appendix A-2)

$$F(\sigma_1, \sigma_2) = F_1(\sigma_1) + F_2(\sigma_2) - \int_0^{\sigma_1} f_1(u) F_2\left(\frac{u}{B_{12}}\right) du - \int_0^{\sigma_2} f_2(v) F_1(B_{12} v)\, dv \tag{2-31a}$$

Equation (2-31a) can be further expanded by integration by parts

$$F(\sigma_1, \sigma_2) = F_1(\sigma_1) + F_2(\sigma_2) - 2F_1(\sigma_1) F_2(\sigma_2) + \frac{1}{B_{12}} \int_0^{\sigma_1} F_1(u)\, f_2\left(\frac{u}{B_{12}}\right) du - B_{12} \int_0^{\sigma_2} F_2(v) f_1(B_{12} v)\, dv \tag{2-31b}$$

Comparing Equations (2-31a) and (2-31b) reveals that the arguments of the pdf and CDF are exchanged so either Equation (2-31a) or (2-31b) can be used depending on simplicity of algebra.

(a) Weibull Distribution Model. If the Weibull function is applied to the independent case, then Equation (2-31a) can be expanded by

$$F(\sigma_1,\sigma_2)=\left[1-\exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1}\right\}\right]+\left[1-\exp\left\{-\left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\}\right]$$

$$-\int_0^{\sigma_1}\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{u}{\beta_1}\right)^{\alpha_1-1}\exp\left\{-\left(\frac{u}{\beta_1}\right)^{\alpha_1}\right\}\left[1-\exp\left\{-\left(\frac{u}{B_{12}\beta_2}\right)^{\alpha_2}\right\}\right]du$$

$$-\int_0^{\sigma_2}\left(\frac{\alpha_2}{\beta_2}\right)\left(\frac{u}{\beta_2}\right)^{\alpha_2-1}\exp\left\{-\left(\frac{u}{\beta_2}\right)^{\alpha_2}\right\}\left[1-\exp\left\{-\left(\frac{B_{12}u}{\beta_1}\right)^{\alpha_1}\right\}\right]du$$

By rearranging terms, the joint distribution function for the Weibull model can be obtained by

$$F(\sigma_1,\sigma_2)=\int_0^{\sigma_1}\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{u}{\beta_1}\right)^{\alpha_1-1}\exp\left\{-\left(\frac{u}{\beta_1}\right)^{\alpha_1}-\left(\frac{u}{B_{12}\beta_2}\right)^{\alpha_2}\right\}du$$

$$+\int_0^{\sigma_2}\left(\frac{\alpha_2}{\beta_2}\right)\left(\frac{u}{\beta_2}\right)^{\alpha_2-1}\exp\left\{-\left(\frac{u}{\beta_2}\right)^{\alpha_2}-\left(\frac{B_{12}u}{\beta_1}\right)^{\alpha_1}\right\}du \qquad \text{(2-32)}$$

(2) <u>Independent And Identical Case</u>. Equation (2-31a) can be further expanded if the identical case is considered.

For the identical case, $F(\sigma) = F_1(\sigma_1) = F_2(\sigma_2)$ and equivalently $f(\sigma) = f_1(\sigma_1) = f_2(\sigma_2)$, so expanding Equation (2-31a)

$$F(\sigma_1,\sigma_2)=F(\sigma_1)+F(\sigma_2)-\int_0^{\sigma_1}f(u)F\left(\frac{u}{B_{12}}\right)du+\int_0^{\sigma_2}f(v)F(B_{12}v)dv \qquad \text{(2-33a)}$$

and similarly Equation (2-31b) can be represented by

$$F(\sigma_1,\sigma_2) = F(\sigma_1) + F(\sigma_2) - 2\,F(\sigma_1)\,F(\sigma_2)$$
$$+\left(\frac{1}{B_{12}}\right)\int_0^{\sigma_1} F(u)\,f\left(\frac{u}{B_{12}}\right)du + B_{12}\int_0^{\sigma_2} F(v)f\left(B_{12}v\right)dv \qquad (2\text{-}33b)$$

For the special case when $B_{12} = 1$, $\sigma = \sigma_1 = \sigma_2$ and Equations (2-33a) and (2-33b) can be simplified to

$$F\left(\sigma_1,\sigma_2\right) = 2F(\sigma) - 2\int_0^{\sigma} f(u)F(u)du$$

and rearranging finally gives the joint distribution function for independent and identical case.(refer to Appendix A-3):

$$F(\sigma_1,\sigma_2) = 2\,F(\sigma) - (F(\sigma))^2 \qquad (2\text{-}34)$$

Equations (2-33a) and (2-33b) can be evaluated for specific models such as the Weibull distribution function.

(a) Weibull Distribution Model. For the Weibull model, the pdf (f(x)) and cumulative density function (CDF: F(x)) can be represented by

$$f(\sigma) = \left(\frac{\alpha}{\beta}\right)\left(\frac{\sigma}{\beta}\right)^{\alpha-1}\exp\left\{-\left(\frac{\sigma}{\beta}\right)^{\alpha}\right\} \qquad (2\text{-}35)$$

$$F(\sigma) = 1 - \exp\left\{-\left(\frac{\sigma}{\beta}\right)^{\alpha}\right\} \qquad (2\text{-}36)$$

Substituting Equation (2-35) and (2-36) into Equation (2-33a)

$$F\left(\sigma_1,\sigma_2\right) = \left[1-\exp\left\{-\left(\frac{\sigma_1}{\beta}\right)^{\alpha}\right\}\right] + \left[1-\exp\left\{-\left(\frac{\sigma_2}{\beta}\right)^{\alpha}\right\}\right]$$
$$-\int_0^{\sigma_1}\left(\frac{\alpha}{\beta}\right)\left(\frac{u}{\beta}\right)^{\alpha-1}\exp\left\{-\left(\frac{u}{\beta}\right)\alpha\right\}\left[1-\exp\left\{-\left(\frac{u}{B_{12}\beta}\right)^{\alpha}\right\}\right]du$$
$$-\int_0^{\sigma_2}\left(\frac{\alpha}{\beta}\right)\left(\frac{v}{\beta}\right)^{\alpha-1}\exp\left\{-\left(\frac{v}{\beta}\right)^{\alpha}\right\}\left[1-\exp\left\{-\left(\frac{B_{12}v}{\beta}\right)^{\alpha}\right\}\right]dv$$

By integrating and rearranging the terms, the joint distribution function of the Weibull model under bi-axial stress σ_1 and σ_2 can be simplified to (refer to Appendix A-4)

$$F(\sigma_1,\sigma_2) = 1 - \exp\left\{-\left(\frac{\sigma_1}{\beta}\right)^{\alpha} - \left(\frac{\sigma_2}{\beta}\right)^{\alpha}\right\} \tag{2-37}$$

D. MEAN FOR THE JOINT DISTRIBUTIONS

1. <u>General Case</u>

If X_1 and X_2 are two continuous random variables having the joint probability density function $f(\sigma_1,\sigma_2)$, the mean or expectation of X_1 and X_2 can be obtained by:

$$\mu_1 = E(\sigma_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_1 f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2 \tag{2-38a}$$

$$\mu_2 = E(\sigma_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_2 f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2 \tag{2-39a}$$

Considering the stress, the bottom limits of the double integration are changed, so Equations (2-38a) and (2-39a) can be expanded specifically for the stress case which we are concerned with

$$\mu_1 = E(\sigma_1) = \int_{0}^{\infty}\int_{0}^{\infty} \sigma_1 f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2 \tag{2-38b}$$

$$\mu_2 = E(\sigma_2) = \int_{0}^{\infty}\int_{0}^{\infty} \sigma_2 f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2 \tag{2-39b}$$

Similarly if three random variables are considered, then Equations (2-38b) and (2-39b) can be expanded by

$$\mu_1 = \int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \sigma_1 f(\sigma_1,\sigma_2,\sigma_6)\,d\sigma_1 d\sigma_2 d\sigma_6 \tag{2-40a}$$

$$\mu_2 = \int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \sigma_2 f(\sigma_1,\sigma_2,\sigma_6)\,d\sigma_1 d\sigma_2 d\sigma_6 \tag{2-40b}$$

$$\mu_6 = \int_0^\infty \int_0^\infty \int_0^\infty \sigma_6 f(\sigma_1, \sigma_2, \sigma_6) d\sigma_1 d\sigma_2 d\sigma_6 \tag{2-40c}$$

2. <u>Specific Case</u>

a. Independent Case

If the random variables X_1 and X_2 are independent for all σ_1 and σ_2

$f(\sigma_1, \sigma_2) = f_1(\sigma_1)\, f_2(\sigma_2)$

and Equations (2-38b) and (2-39b) can be expanded by

$$\mu_1 = \int_0^\infty \int_0^\infty \sigma_1 f_1(\sigma_1) f_2(\sigma_2) d\sigma_1 d\sigma_2 \tag{2-41a}$$

$$\mu_2 = \int_0^\infty \int_0^\infty \sigma_2 f_1(\sigma_1) f_2(\sigma_2) d\sigma_1 d\sigma_2 \tag{2-42a}$$

taking the integral and noting that $F_1(\infty) = F_2(\infty) = 1$, then Equations (2-41a) and (2-42a) can be simplified to

$$\mu_1 = \int_0^\infty \sigma_1 f_1(\sigma_1) d\sigma_1 \tag{2-41b}$$

$$\mu_2 = \int_0^\infty \sigma_2 f_2(\sigma_2) d\sigma_2 \tag{2-42b}$$

Similarly, for the three random variable case:

$$\mu_1 = \int_0^\infty \int_0^\infty \int_0^\infty \sigma_1 f_1(\sigma_1) f_2(\sigma_2) f_6(\sigma_6) d\sigma_1 d\sigma_2 d\sigma_6 = \int_0^\infty \sigma_1 f_1(\sigma_1) d\sigma_1 \tag{2-43a}$$

$$\mu_2 = \int_0^\infty \int_0^\infty \int_0^\infty \sigma_2 f_1(\sigma_1) f_2(\sigma_2) f_6(\sigma_6) d\sigma_1 d\sigma_2 d\sigma_6 = \int_0^\infty \sigma_2 f_2(\sigma_2) d\sigma_2 \tag{2-43b}$$

$$\mu_6 = \int_0^\infty \int_0^\infty \int_0^\infty \sigma_6 f_1(\sigma_1) f_2(\sigma_2) f_6(\sigma_6) d\sigma_1 d\sigma_2 d\sigma_6 = \int_0^\infty \sigma_6 f_6(\sigma_6) d\sigma_6 \tag{2-43c}$$

Observing Equations (2-41b) through (2-43c), it can be concluded that the mean values of the random variables X_1, X_2, and X_6 are dependent only on their corresponding probability density functions.

b. Independent And Identical Case

For the independent and identical case, the joint probability function can be obtained by

$$f(\sigma_1,\sigma_2,\sigma_6) = f(\sigma_1)\, f(\sigma_2)\, f(\sigma_6)$$

where $f(\sigma) = f_1(\sigma) = f_2(\sigma) = f_6(\sigma)$

Applying these relations to Equations (2-41b) through (2-43c), the mean values for the joint distributions can be simplified for the independent and identical case.

$$\mu_1 = \mu_2 = \mu_6 = \int_0^{\infty} \sigma f(\sigma) d\sigma \tag{2-44}$$

(1) Weibull Distribution Model. If the Weibull probability function is considered, Equation (2-44) can be expanded as follows

$$\mu = \int_0^{\infty} \sigma\left(\frac{\alpha}{\beta}\right)\left(\frac{\sigma}{\beta}\right)^{\alpha-1} \exp\left\{-\left(\frac{\sigma}{\beta}\right)^{\alpha}\right\} d\sigma$$

letting $= (\sigma/\beta)^a$, then

$$\mu = \int_0^{\infty} \beta y^{\frac{1}{\alpha}} \exp(-y) dy$$

Using the definition of the Gamma function, we finally get the mean function:

$$\mu = \beta\Gamma\left(1 + \frac{1}{\alpha}\right) \tag{2-45}$$

E. JOINT RELIABILITY

1. Concept Of Reliability

The common notion of reliability is the confidence in the ability of a device to perform adequately over a range of specified conditions. Such qualitative measurement of confidence and adequacy can be quantitatively calculated by probability. In the formalism of probability, the conditions over which the device is intended to perform is defined as the random variable X: and the probability over the range of conditions is defined as the CDF over the range that the random variable experiences in service. That is, over the sample space σ experienced in service, the probability of failure is defined by the CDF over the sample space:

$$P(X \leq \sigma) = F(\sigma)$$

Since reliability is the compliment of failure, it is defined by

$$R(\sigma) = 1 - F(\sigma) \tag{2-46}$$

2. Joint Reliability

If X_1 and X_2 are two random variable, then the joint reliability of X_1 and X_2 can be defined by

$$R(\sigma_1, \sigma_2) = 1 - P(X_1 \leq \sigma_1, X_2 \leq \sigma_2) \tag{2-47}$$

or equivalently for the three random variable case

$$R(\sigma_1, \sigma_2, \sigma_6) = 1 - P(X_1 \leq \sigma_1, X_2 \leq \sigma_2, X_6 \leq \sigma_6)$$

For the discrete case, equation (2-47) can be expanded by:

$$\begin{aligned} R(\sigma_1, \sigma_2) &= 1 - \sum_{u \leq \sigma_1} \sum_{v \leq \sigma_2} f(u, v) \\ R(\sigma_1, \sigma_2, \sigma_6) &= 1 - \sum_{u \leq \sigma_1} \sum_{v \leq \sigma_2} \sum_{w \leq \sigma_6} f(u, v, w) \end{aligned} \tag{2-48}$$

If the continuous case is considered, the joint reliability function of X_1 and X_2 is defined by

$$R(\sigma_1, \sigma_2) = 1 - \int_{-\infty}^{\sigma_1} \int_{-\infty}^{\sigma_2} f(u, v) dv du \tag{2-49}$$

and from Equation (2-49) the marginal reliability function can be obtained:

$$R_1(\sigma_1) = 1 - \int_{-\infty}^{\sigma_1}\int_{-\infty}^{\infty} f(u,v)\,dv\,du \tag{2-50a}$$

$$R_2(\sigma_2) = 1 - \int_{-\infty}^{\sigma_2}\int_{-\infty}^{\infty} f(u,v)\,du\,dv \tag{2-50b}$$

If the three random variable cases are considered, Equation (2-50a) and (2-50b) can be further expanded:

$$R_1(\sigma_1) = 1 - \int_{-\infty}^{\sigma_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v,w)\,dv\,dw\,du \tag{2-51a}$$

$$R_2(\sigma_2) = 1 - \int_{-\infty}^{\sigma_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v,w)\,dw\,du\,dv \tag{2-51b}$$

$$R_6(\sigma_6) = 1 - \int_{-\infty}^{\sigma_6}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v,w)\,du\,dv\,dw \tag{2-51c}$$

$R_1(\sigma_1)$, $R_2(\sigma_2)$, and $R_6(\sigma_6)$ are also equivalent to the one dimensional reliability function which are defined by

$$R_1(\sigma_1) = 1 - \int_{-\infty}^{\sigma_1} f_1(u)\,du \tag{2-52a}$$

$$R_2(\sigma_2) = 1 - \int_{-\infty}^{\sigma_2} f_2(u)\,du \tag{2-52b}$$

$$R_6(\sigma_6) = 1 - \int_{-\infty}^{\sigma_6} f_6(u)\,du \tag{2-52c}$$

a. General Case

The general equations for reliability can now be applied to the calculation of the mechanical strength reliability of composites under combined stresses by specifying the random variables as X_i and the realized random variables as σ_i. The joint reliability under combined stress σ_1 and σ_2 can be obtained by

$$R(\sigma_1,\sigma_2) = 1 - F(\sigma_1,\sigma_2) = 1 - P\{X = x_1 \wedge x_2 \text{over}(\sigma_1,\sigma_2)\}$$

where X_1 and X_2 are the failure strength in each coordinate(see Figure 2-2).

Noting that the joint pdf under combined stress σ_1 and σ_2 is represented by the double integration of $f(\sigma_1, \sigma_2)$ over the domains S_1 and S_2 and that the double integration of $f(\sigma_1, \sigma_2)$ over the domain S_3 (see Figure 2-2) is defined as the joint reliability under combined stress σ_1 and σ_2, then the following relation can be obtained:

$$R(\sigma_1,\sigma_2) = 1 - F(\sigma_1,\sigma_2) = \int_{\sigma_2}^{\infty}\int_{\sigma_1}^{\infty} f(u,v)dudv \tag{2-53}$$

Comparing Equation (2-53) to Equation (2-26), Equation (2-53) can be represented in another form:

$$R(\sigma_1,\sigma_2) = 1 - \int_0^{\sigma_1}\int_{\frac{u}{B_{12}}}^{\infty} f(u,v)dvdu + \int_0^{\sigma_2}\int_{B_{12}v}^{\infty} f(u,v)dudv \tag{2-54}$$

If the joint reliability under three combined stress is considered, equation (2-53) can be expanded by

$$R(\sigma_1,\sigma_2,\sigma_6) = 1 - F(\sigma_1,\sigma_2,\sigma_6) = \int_{\sigma_6}^{\infty}\int_{\sigma_2}^{\infty}\int_{\sigma_1}^{\infty} f(u,v,w)dudvdw \tag{2-55}$$

b. Specific Case

The joint reliability of bi-axial stress case can be further expanded for the specific case as in the probability distribution function.

(1) Independent Case. If the random variable X_1 and X_2 are independent for all σ_1 and σ_2, then Equation (2-53) can be expanded:

$$R(\sigma_1,\sigma_2) = \int_{\sigma_2}^{\infty}\int_{\sigma_1}^{\infty} f_1(u)f_2(v)dudv \tag{2-56}$$

After integration, Equation (2-56) can be simplified to (refer to Appendix A-5)

$$R(\sigma_1,\sigma_2) = \{1-F_1(\sigma_1)\}\{1-F_2(\sigma_2)\} \tag{2-57}$$

noting that $R_1(\sigma_1) = 1-F_1(\sigma_1)$ and $R_2(\sigma_2) = 1-F_2(\sigma_2)$

$$R(\sigma_1,\sigma_2) = R_1(\sigma_1)\, R_2(\sigma_2) \tag{2-58}$$

For the three random variable case, Equations (2-56), (2-57), and (2-58) can be obtained by

$$R(\sigma_1,\sigma_2,\sigma_6) = \int_{\sigma_6}^{\infty}\int_{\sigma_2}^{\infty}\int_{\sigma_1}^{\infty} f_1(u)\, f_2(v)\, f_6(w)\,du\,dv\,dw$$

$$= (1-F_1)\,(1-F_2)\,(1-F_6)$$

$$= R_1(\sigma_1)\, R_2(\sigma_2)\, R_6(\sigma_6) \tag{2-59}$$

In application to a composite, if the probability of failure model (F_i) is known, then the reliability of the composite can be calculated. The probability of failure model can be inferred from the physical consideration of the failure processes.

When a filament composite is loaded along the fiber direction, local failure begins when the weakest fiber fails. The load carried by the broken fiber is transferred to the neighboring fibers. Upon additional increase in load, additional fibers fail, leading to the increase of failure sites distributed over the composite. The higher the load, the higher the density of such failure sites and the higher the probability of clustering. The spatial clustering of the fiber failure sites leads to stress concentration and ultimately causes the catastrophic failure of the composite. Harlow and Phoenix investigated the probabilistic modeling of the above sequential failure events and arrived at a modified weakest link model in which the link of the chain is a bundle. [Ref. 2] For a limited range of the random variable, it can be approximated by the two parameter Weibull model [Ref. 3]:

$$R(\sigma_1) = \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1}\right\}$$

When a filament composite is loaded perpendicular to the fiber direction, the failure mode is similar to that described by classical fracture mechanics. The exception is that for a composite, all inherent flaws are not randomly oriented, but are aligned with and propagate along the fiber. In such a case, the largest crack dominates and it is effectively a weakest link process over the physical volume. On these physical grounds we may use the Weibull model for both the transverse strength and shear strength:

$$R(\sigma_2) = \exp\left\{-\left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\}$$

$$R(\sigma_6) = \exp\left\{-\left(\frac{\sigma_6}{\beta_6}\right)^{\alpha_6}\right\}$$

Equation (2-57) can be specified for these Weibull distribution functions:

$$R(\sigma_1,\sigma_2) = \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} - \left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\} \qquad (2\text{-}60)$$

equivalently for the three random variable case:

$$R(\sigma_1,\sigma_2,\sigma_6) = \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} - \left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2} - \left(\frac{\sigma_6}{\beta_6}\right)^{\alpha_6}\right\} \qquad (2\text{-}61)$$

If we note $F(\sigma_1, \sigma_2) = 1 - R(\sigma_1, \sigma_2)$, then the joint distribution function obtained in Equation (2-32) can be simplified by

$$F(\sigma_1,\sigma_2) = 1 - \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} - \left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\} \qquad (2\text{-}62)$$

$$F(\sigma_1,\sigma_2,\sigma_6) = 1 - \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} - \left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2} - \left(\frac{\sigma_6}{\beta_6}\right)^{\alpha_6}\right\} \qquad (2\text{-}63)$$

(2) Independent And Identical Case. If the random variables X_1 and X_2 are independent for all σ_1 and σ_2 and the probability distribution function for

each random variable is the same, then $F(\sigma) = F_1(\sigma_1) = F_2(\sigma_2)$. Similarly $R(\sigma) = R_1(\sigma_1) = R_2(\sigma_2)$ and the Equation (2-58) can be simplified to

$$R(\sigma_1, \sigma_2) = R(\sigma_1)\, R(\sigma_2) \tag{2-64}$$

As a specific case, if the Weibull distribution function is considered, Equation (2-64) can be further expanded:

$$R(\sigma_1, \sigma_2) = \exp\left\{ -\left(\frac{\sigma_1}{\beta}\right)^{\alpha} - \left(\frac{\sigma_2}{\beta}\right)^{\alpha} \right\} \tag{2-65}$$

For $B_{12} = 1$, then $\sigma = \sigma_1 = \sigma_2$ and the joint reliability function for Weibull model is obtained.

$$R(\sigma, \sigma) = \exp\left\{ -2\left(\frac{\sigma}{\beta}\right)^{\alpha} \right\} \tag{2-66}$$

For the three random variable case, Equations (2-64) through (2-66) can be expanded by

$$R(\sigma_1, \sigma_2, \sigma_6) = R(\sigma_1)\, R(\sigma_2)\, R(\sigma_6) \tag{2-67}$$

$$R(\sigma_1, \sigma_2, \sigma_6) = \exp\left\{ -\left(\frac{\sigma_1}{\beta}\right)^{\alpha} - \left(\frac{\sigma_2}{\beta}\right)^{\alpha} - \left(\frac{\sigma_6}{\beta}\right)^{\alpha} \right\} \tag{2-68}$$

$$R(\sigma, \sigma, \sigma) = \exp\left\{ -3\left(\frac{\sigma}{\beta}\right)^{\alpha} \right\} \tag{2-69}$$

III. GEOMETRIC REPRESENTATION OF JOINT RELIABILITY UNDER COMBINED STRESS

A. BACKGROUND FOR GEOMETRIC REPRESENTATION

CDF and pdf are useful functions which contain all the relevant information about the statistical properties of a random variable, in our case, the strength of composite. These functions are required in reliability analysis, design, acceptance, maintenance, and operational logistics.

In applications, in order to map out the statistical failure surface even for the bivariate case (the biaxial combined stress), an exceedingly large number of experiments are required. A large number of experiments are frequently impractical due to economic and time constrains and other considerations. Therefore, it is important to visualize the shape of the failure surface in order to narrow the range of experiments to be focused on the critical regions in the stress domain.

In this chapter we present the geometric representations for the reliability functions for the independent case derived in Chapter II. The graphical presentation is based on the probability plots and the failure surface representations, which were also investigated. Examination of these graphical representations will shed light on the appropriate experiments necessary for the identification of whether the failure processes are independent or dependent. When the independence is established, the entire failure surface (all permutations and combinations of combined stress case) can be calculated from the uniaxial strength statistics.

B. JOINT RELIABILITY FUNCTION

1. <u>Two Random Variable Case</u>

From Equations (2-56) and (2-66), the reliability function for the Weibull model can be obtained by

$$R(\sigma) = \exp\left\{-\left(\frac{\sigma}{\beta}\right)^{\alpha}\right\} \tag{3-1}$$

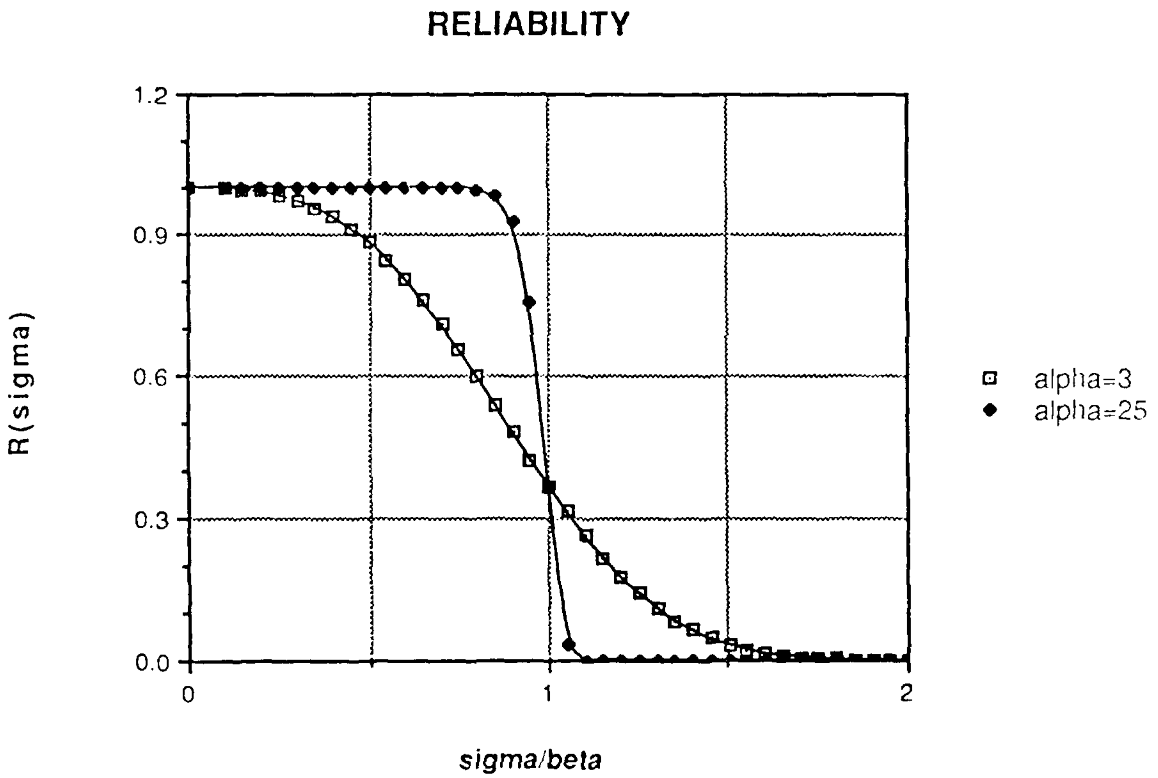

Figure 3-1 . Effects Of α In Reliability

If we consider σ/β as a normalized variable, then the reliability function versus σ/β can be plotted for various values of α. Figure 3-1 shows two different graphes for large and small values of α. For the small values of α, the graph shows a relatively smooth curve, but as α increases, the graph approaches to the step function and these relations can be further expanded for the joint reliability problem. If the independent combined stress cases are considered, the joint reliability of the composite under combined stress can be obtained from Equation (2-58).

$$R(\sigma_1,\sigma_2) = R_1(\sigma_1)\, R_2(\sigma_2)$$

Expanding $R(\sigma_1, \sigma_2)$

$$R(\sigma_1,\sigma_2) = \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} - \left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\} \tag{2-60}$$

where

$$R(\sigma_1) = \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1}\right\}, R(\sigma_2) = \exp\left\{-\left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\}$$

If we assume that

$$\beta_1 = V_{12}\, \beta_2 \tag{3-2}$$

and noting that

$$\sigma_1 = B_{12}\, \sigma_2$$

then the joint reliability for the independent case can be represented by

$$R(\sigma_1,\sigma_2) = R_1(\sigma_1)\, R_2(\sigma_2)$$

$$= \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1}\right\} \exp\left\{-\left(\frac{V_{12}}{B_{12}}\frac{\sigma_1}{\beta_1}\right)^{\alpha_2}\right\} \tag{3-3}$$

If we further assume $C_{12}=V_{12}/B_{12}$, then Equation (3-3) can be simplified by

$$R(\sigma_1,\sigma_2) = \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1}\right\}\exp\left[-\left\{C_{12}\left(\frac{\sigma_1}{\beta_1}\right)\right\}^{\alpha_2}\right] \tag{3-4}$$

From Equation (3-4), the reliability functions $R_1(\sigma_1)$ and $R_2(\sigma_2)$ can be plotted independently in two dimensional space provided that α_1, α_2, and C_{12} are given. Then $R_1(\sigma_1)$ and $R_2(\sigma_2)$ are combined to yield the joint reliability function of $R(\sigma_1,\sigma_2)$ as shown Figure 3-2. Observing the Figure 3-2, it can be noted that the joint reliability is affected by the smaller value between $R_1(\sigma_1)$ and $R_2(\sigma_2)$ for a given value of σ_1/β_1, that is, the value of the joint reliability is always close to the smaller value between $R_1(\sigma_1)$ and $R_2(\sigma_2)$. When the values of both $R_1(\sigma_1)$ and $R_2(\sigma_2)$ are close to '1' for the specific range of σ_1/β_1, the value of the joint reliability is also close to 1, but as either $R_1(\sigma_1)$ or $R_2(\sigma_2)$ decreases, the joint reliability also decrease depending on the smaller value between R_1 and R_2. We can also observe the effect of C_{12} for the given α_1 and α_2.

As shown in Figures 3-2a and 3-2c, the $R_2(\sigma_2)$ shifts to the right from the $R_1(\sigma_1)$ as C_{12} decreases from '1' whereas $R_2(\sigma_2)$ shifts to the left as C_{12} increases from '1' while $R_1(\sigma_1)$ remains constant. So, the joint reliability follows the smaller function between $R_1(\sigma_1)$ and $R_2(\sigma_2)$ and appears to almost overlap when C_{12} is far from '1'. As the value of C_{12} approaches to '1', the values of $R_1(\sigma_1)$ and $R_2(\sigma_2)$ approach each other and when C_{12}=1, $R_1(\sigma_1)$ and $R_2(\sigma_2)$ cross each other at σ_1/β_1=1. The joint reliability function exists to the left of $R_1(\sigma_1)$ and $R_2(\sigma_2)$ and has a weaker reliability than $R_1(\sigma_1)$ or $R_2(\sigma_2)$. So the joint reliability is always less than other uni-axial directional reliabilities.

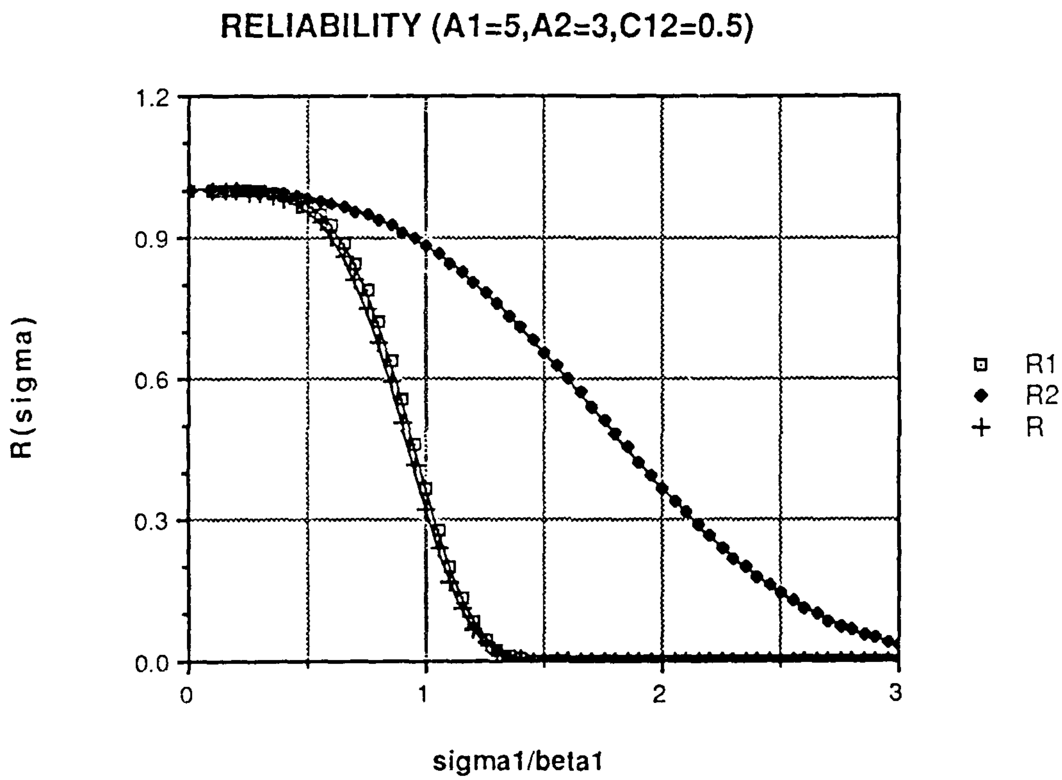

Figure 3-2a : Reliability Vs σ_1/β_1 (C_{12}=0.5)

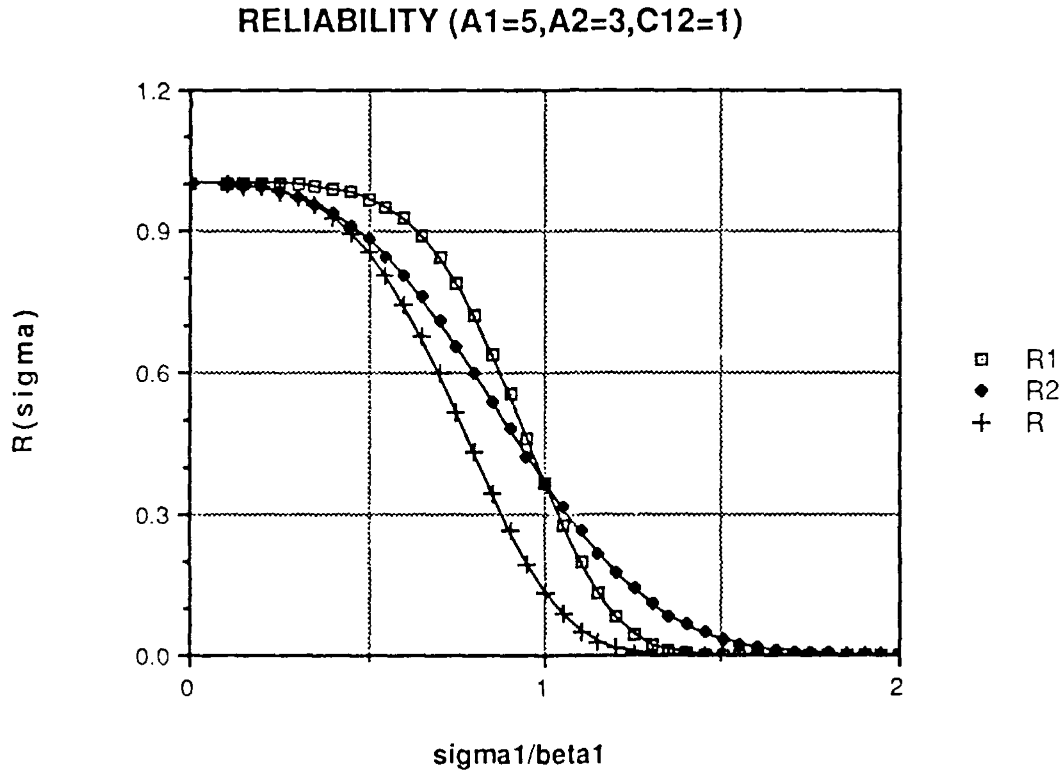

Figure 3-2b : Reliability Vs σ_1/β_1 (C_{12}=1)

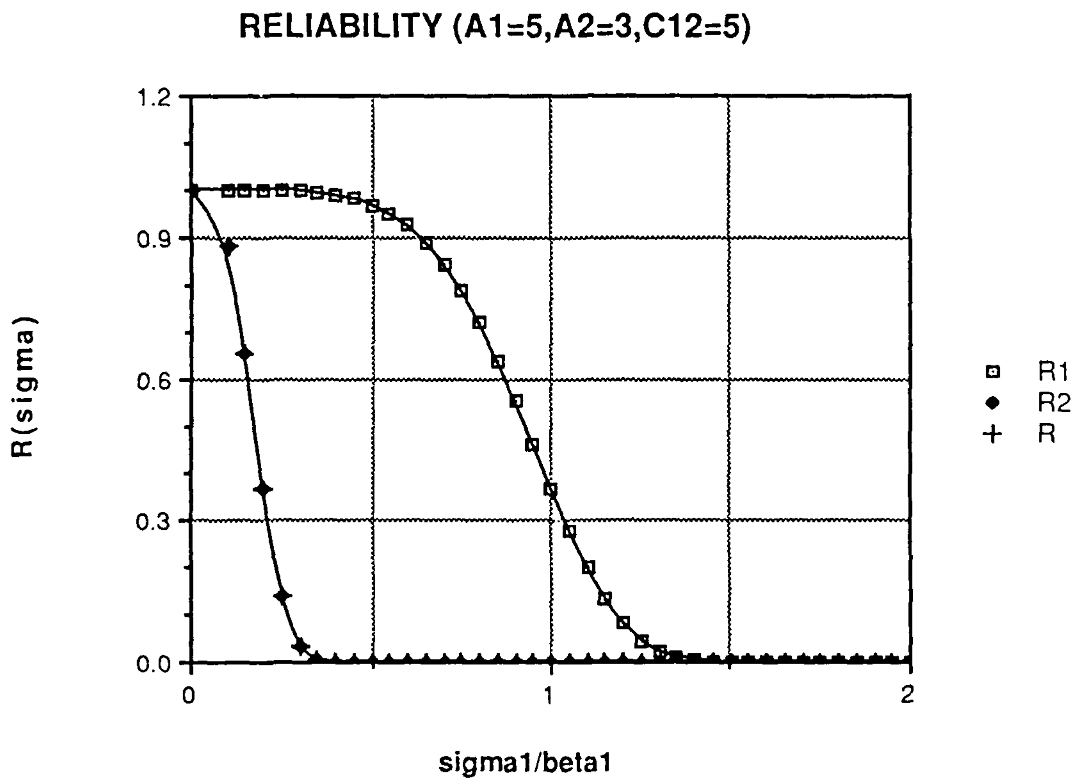

Figure 3-2c : Reliability Vs σ_1/β_1 (C_{12}=5)

Figure 3-3 shows the joint reliability in three dimensional space for different values of C_{12} and for fixed α_1 and α_2.

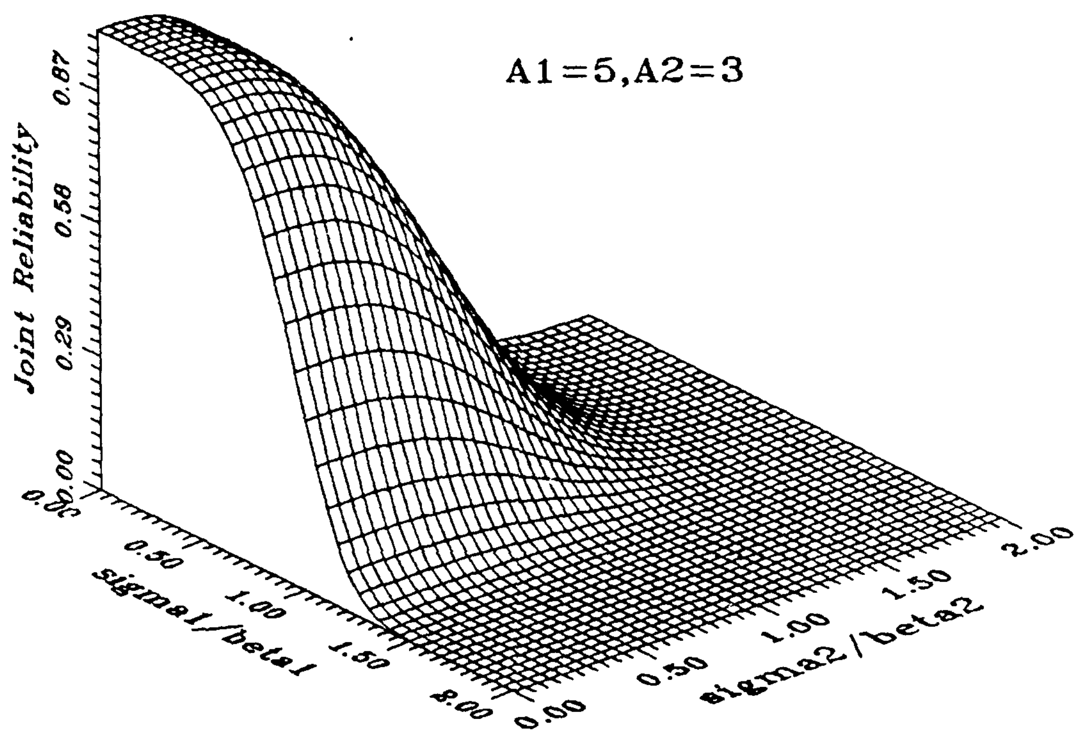

Figure 3-3 : Reliability In 3-D Space (α_1=5, α_2=3)

Here we introduce the concept of linearized failure CDF to show the effect of the joint failure CDF at the tail area.

To get the linearized failure CDF, denoted by $F^*(\sigma)$, use logarithmic algebra. If we take the logarithm of $R_1(\sigma_1)$ and note that $R_1(\sigma_1) = 1 - F_1(\sigma_1)$, then

$$\ln\{R_1(\sigma_1)\} = \ln\{1 - F_1(\sigma_1)\} = -\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} \tag{3-5}$$

Multiplying both side by (-1) and taking the logarithm again, the linearized failure CDF can then be defined as

$$F_1^*(\sigma_1) = \ln\left[-\ln\{R_1(\sigma_1)\}\right] \tag{3-6}$$

Substituting Equation (3-5) into Equation (3-6), then

$$F_1^*(\sigma_1) = \alpha_1 \ln\left(\frac{\sigma_1}{\beta_1}\right) \tag{3-7}$$

By the same way, linearized failure CDF corresponding to σ_2 can be obtained:

$$F_2^*(\sigma_2) = \alpha_2\left\{\ln(C_{i2}) + \ln\left(\frac{\sigma_1}{\beta_1}\right)\right\} \tag{3-8}$$

Comparing Equation (3-8) to (3-7), it can be noted that $F_2^*(\sigma_2)$ is vertically shifted and rotated when compared to $F_1^*(\sigma_1)$. This result is represented in Figure 3-4.

Similarly $F^*(\sigma_1,\sigma_2)$ can be obtained by

$$F^*(\sigma_1,\sigma_2) = \ln\left\{\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} + \left(C_{12}\frac{\sigma_1}{\beta_1}\right)^{\alpha_2}\right\} \tag{3-9}$$

Comparing Equation (3-9) to (3-7) and (3-8), it can be noted that $F^*(\sigma_1,\sigma_2)$ is no longer a linear relation in terms of $\ln(\sigma_1/\beta_1)$. Figures 3-4a, 3-4b, and 3-4c show the linearized failure CDF for different values of C_{12}. As shown in these graphs, $F_2^*(\sigma_2)$ is shifted upward and rotated clockwise as C_{12} increases and this causes a higher probability of failure. The joint linearized failure CDF, $F^*(\sigma_1,\sigma_2)$, then follows the higher value between $F_1^*(\sigma_1)$ and $F_2^*(\sigma_2)$ and does not show the linear line, especially near the area where $F_1^*(\sigma_1)$ and $F_2^*(\sigma_2)$ intersect each other. So we can observe that , as the value of C_{12} increases for the specified α_1 and α_2, $F_2^*(\sigma_2)$ shifts upward and rotates clockwise causing the intersection between $F_1^*(\sigma_1)$ and $F_2^*(\sigma_2)$ to go downward, or in other words creating a higher probability of failure.

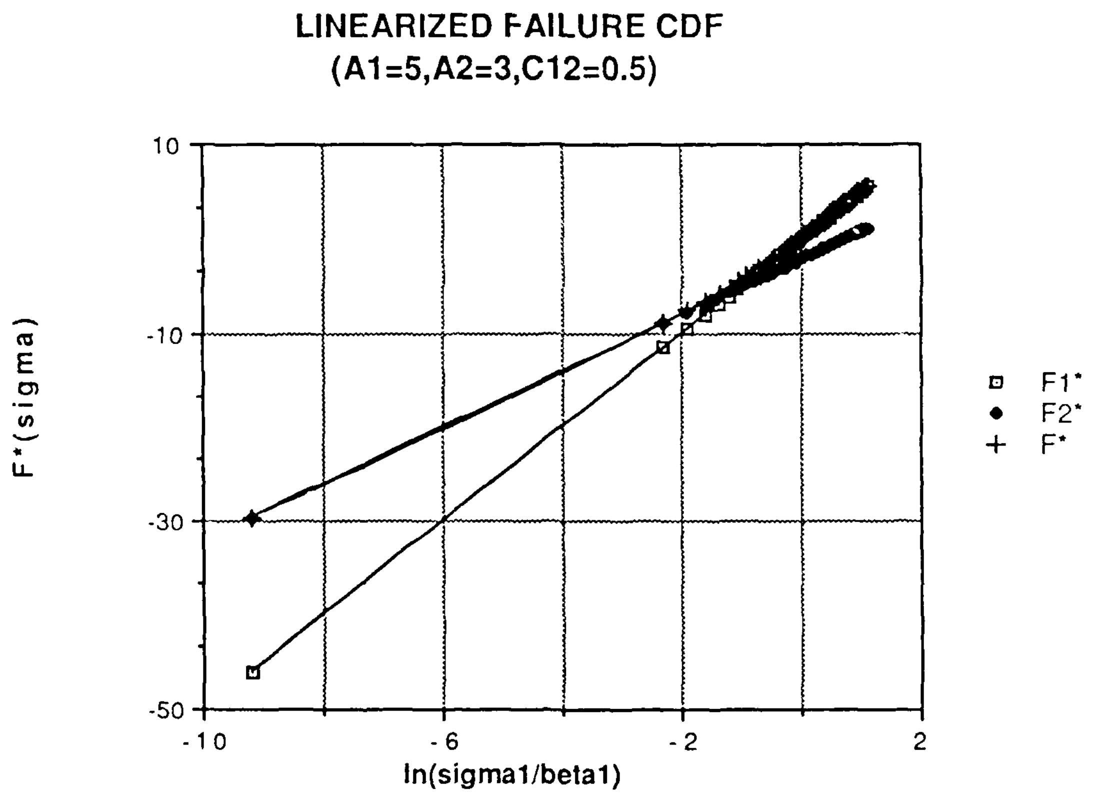

Figure 3-4a : Linearized Failure CDF (C_{12}=0.5)

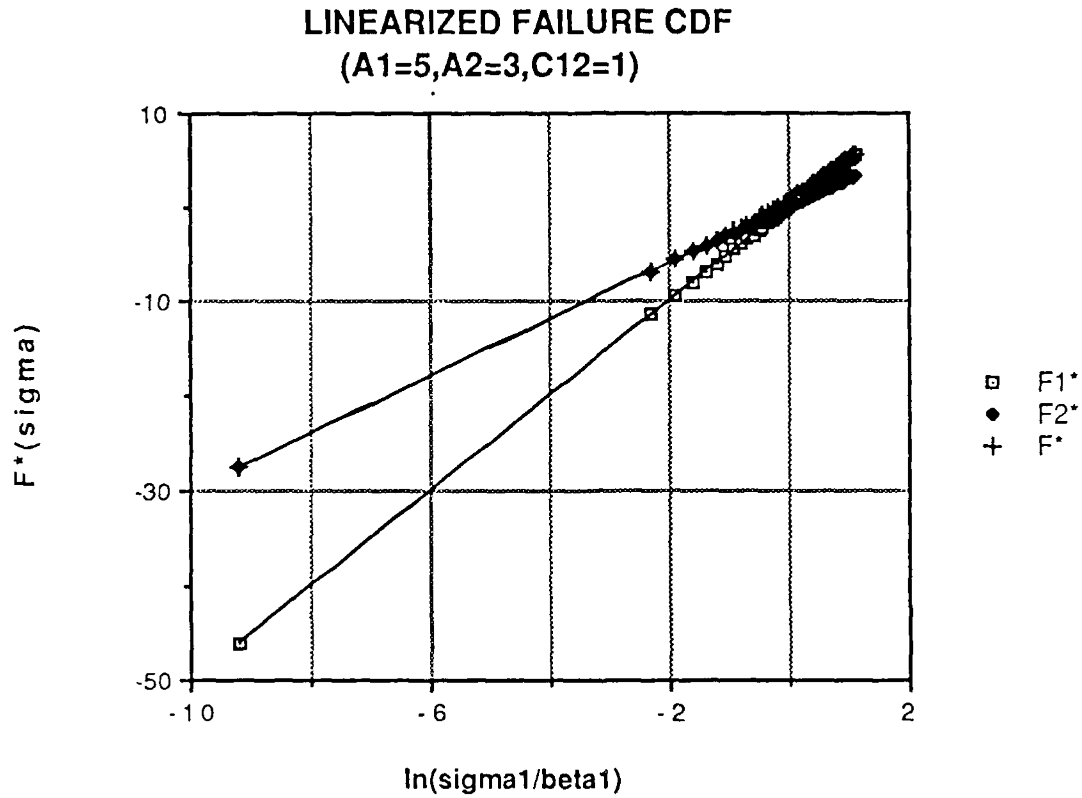

Figure 3-4b : Linearized Failure CDF (C_{12}=1)

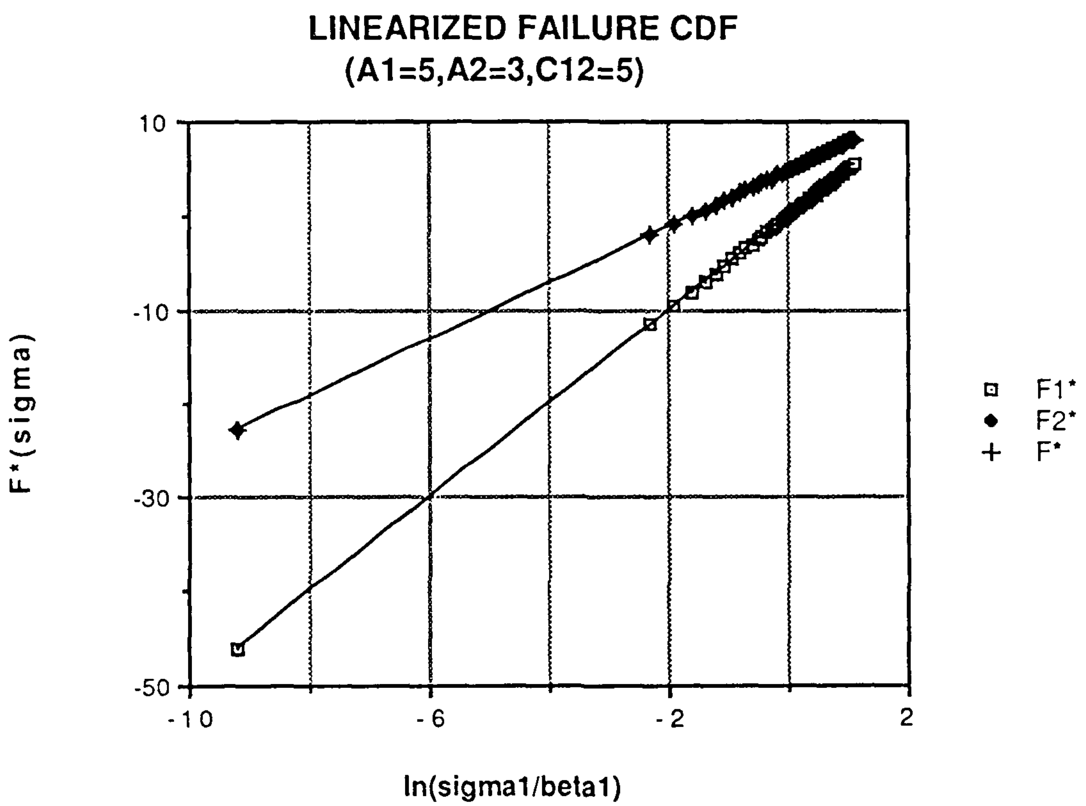

Figure 3-4c : Linearized Failure CDF (C_{12}=5)

As a special case, if $\alpha=\alpha_1=\alpha_2$, then the joint reliability function and Equations (3-7) through (3-9) can be simplified by

$$R(\sigma_1,\sigma_2) = \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha}\left(1 + C_{12}^{\alpha}\right)\right\} \tag{3-10}$$

$$F_1^*(\sigma_1) = \alpha \ln\left(\frac{\sigma_1}{\beta_1}\right) \tag{3-11}$$

$$F_2^*(\sigma_2) = \alpha\left\{\ln\left(C_{12}\right) + \ln\left(\frac{\sigma_1}{\beta_1}\right)\right\} \tag{3-12}$$

$$F^*(\sigma_1,\sigma_2) = \alpha \ln\left(\frac{\sigma_1}{\beta_1}\right) + \ln\left(1 + C_{12}^{\alpha}\right) \tag{3-13}$$

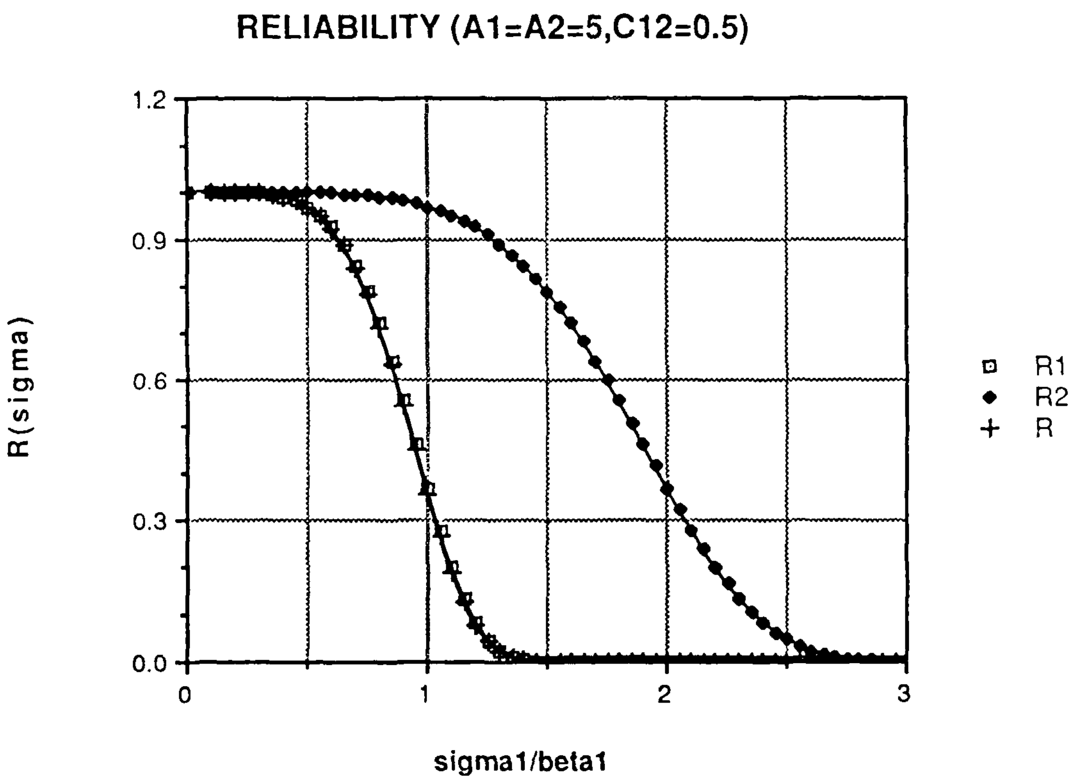

Figure 3-5a : Reliability Vs $\sigma_1/\beta_1(\alpha_1=\alpha_2=5)$

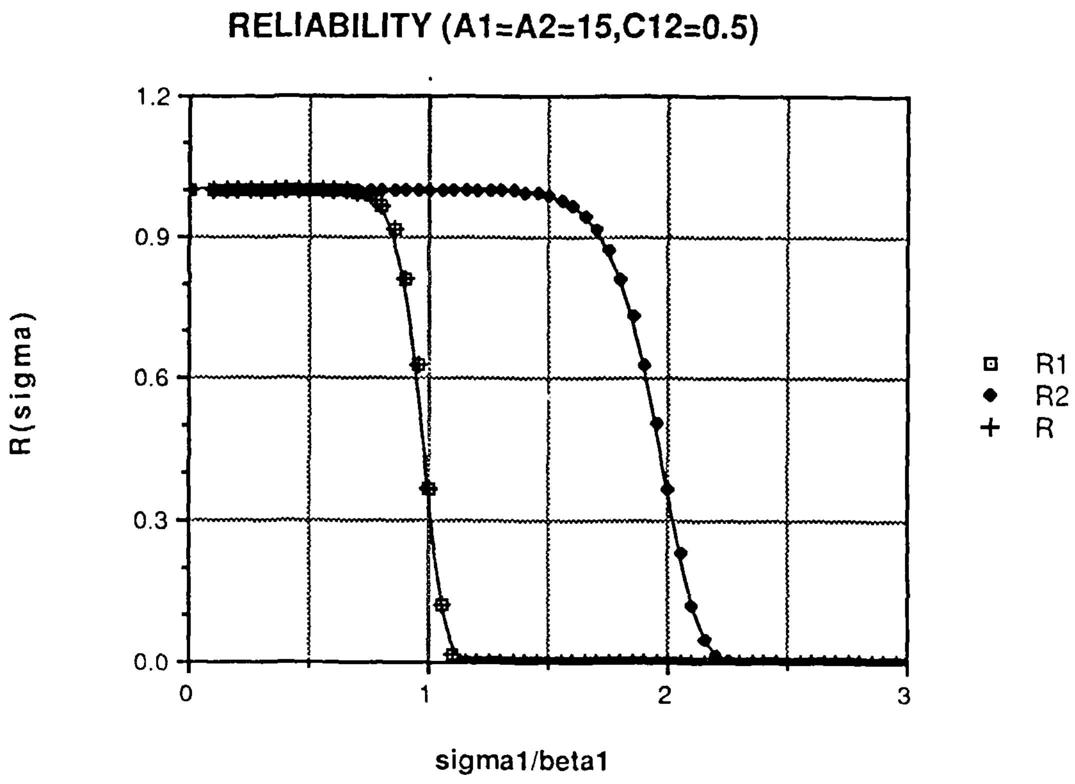

Figure 3-5b : Reliability Vs σ_1/β_1 ($\alpha_1=\alpha_2=15$)

Comparing Equations (3-12) and (3-13) to Equation (3-10), it can be noted that $F_2^*(\sigma_2)$ is shifted vertically by $\alpha \ln(v_{12}/B_{12})$. $F^*(\sigma_1,\sigma_2)$ is also shifted vertically by $\ln\{1+(v_{12}/B_{12})\}$ regardless of values of C_{12}. Figures 3-5 though 3-7 show the reliability in two and three dimensional space and linearized failure CDF in two dimensional space for $\alpha = \alpha_1 = \alpha_2$.

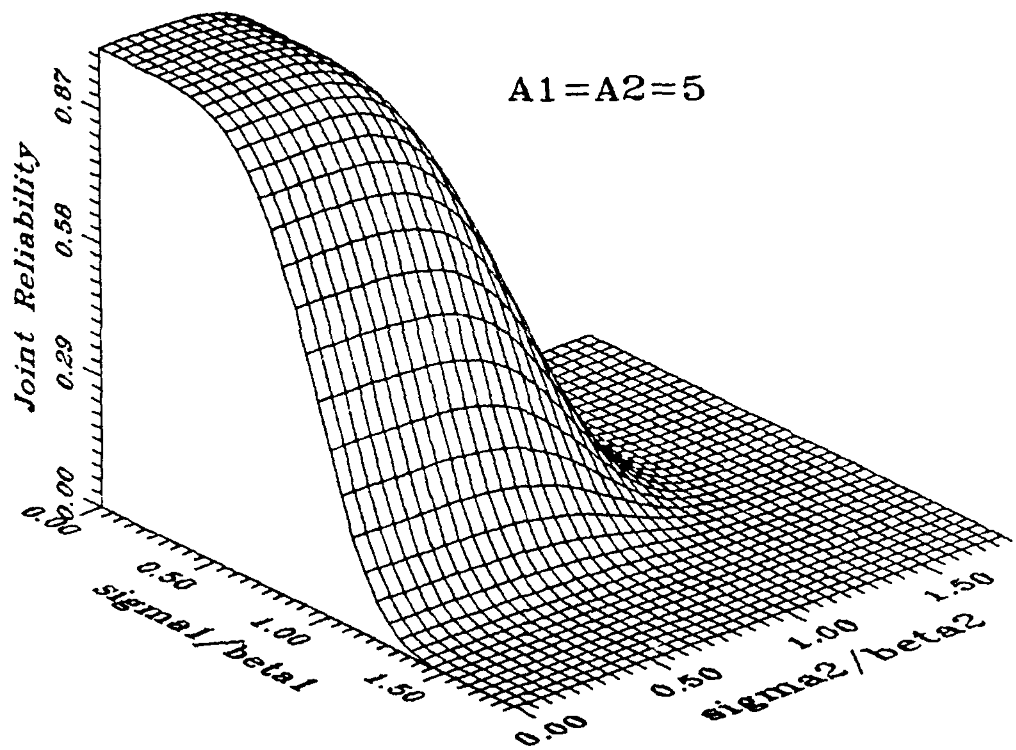

Figure 3-6a : Joint Reliability In 3-D (α_1=α_2=5)

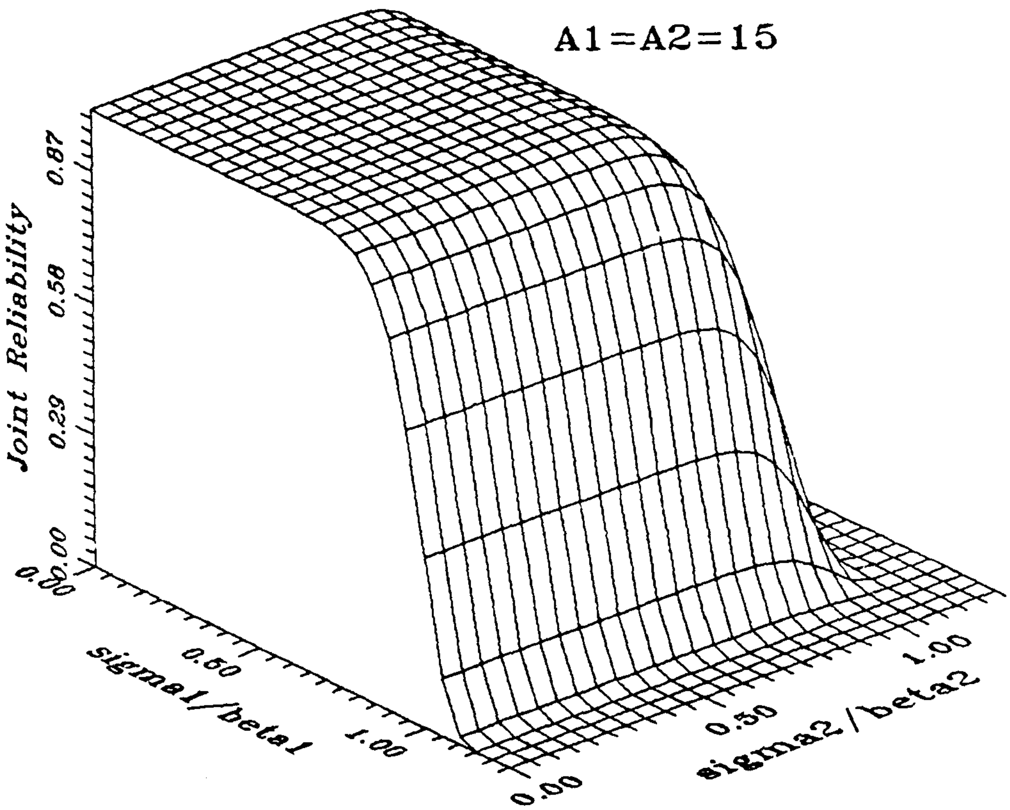

Figure 3-6b : Joint Reliability In 3-D (α_1=α_2=15)

As shown in Figure 3-7, each failure CDF is parallel to each other and the joint linearized failure CDF almost coincides with $F_1^*(\sigma_1)$ or $F_2^*(\sigma_2)$, depending on the value of C_{12}, and is parallel to both $F_1^*(\sigma_1)$ and $F_2^*(\sigma_2)$. So in the physical sense, C_{12} is very important for estimating the failure and reliability. The joint reliability is affected by the weakest reliability function and this is true for the joint linearized failure CDF.

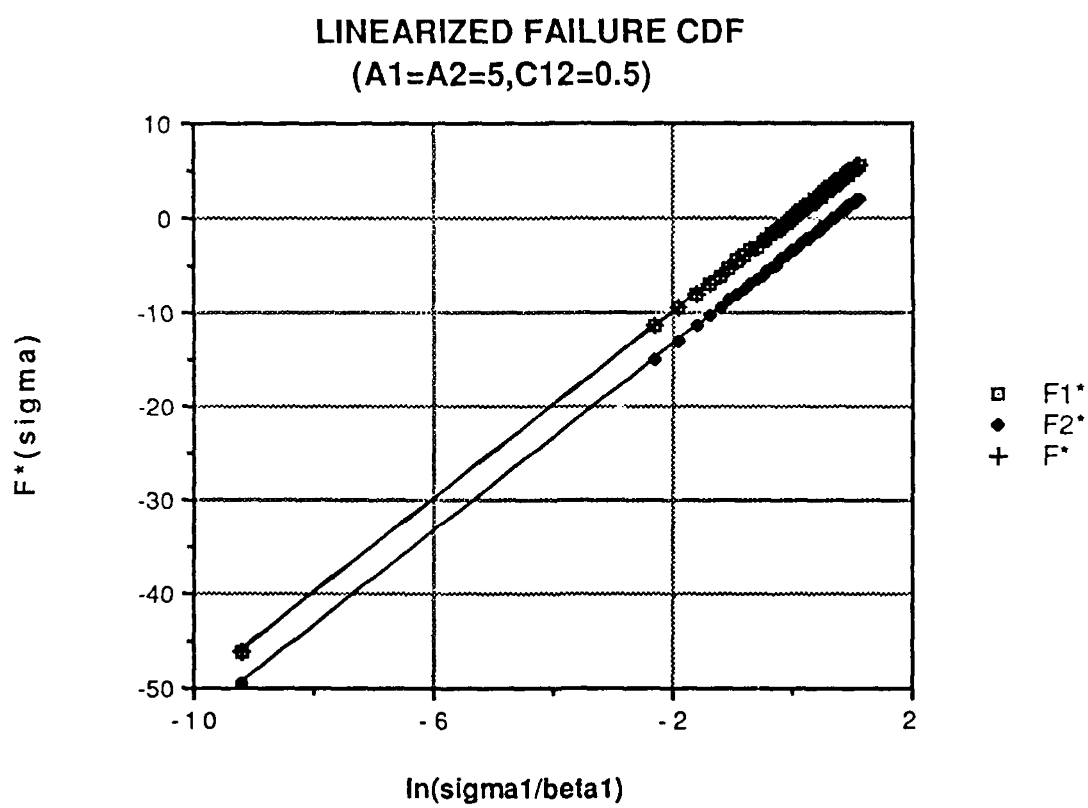

Figure 3-7a : Linearized Failure CDF ($\alpha_1=\alpha_2=5$)

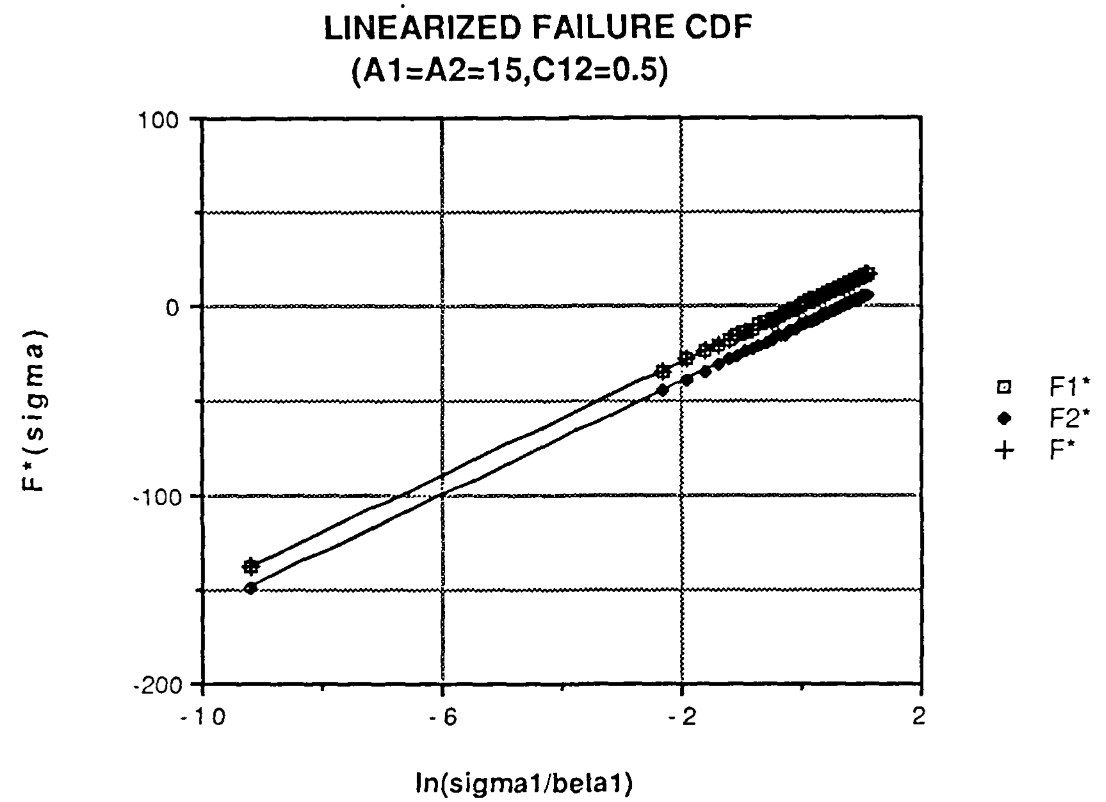

Figure 3-7b: Linearized Failure CDF ($\alpha_1=\alpha_2=15$)

2. <u>Three Random Variable Case</u>

If the three random variable case is considered, the joint reliability under the combined stress can be obtained from the Equation (2-79):

$$R(\sigma_1, \sigma_2, \sigma_6) = R_1(\sigma_1)\, R_2(\sigma_2)\, R_6(\sigma_6)$$

Expanding this equation for the Weibull model:

$$R(\sigma_1,\sigma_2,\sigma_6) = \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} - \left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2} - \left(\frac{\sigma_6}{\beta_6}\right)^{\alpha_6}\right\} \qquad (2\text{-}61)$$

RELIABILITY (A1=25,A2=5,A6=3)
(C12=0.5,C16=1)

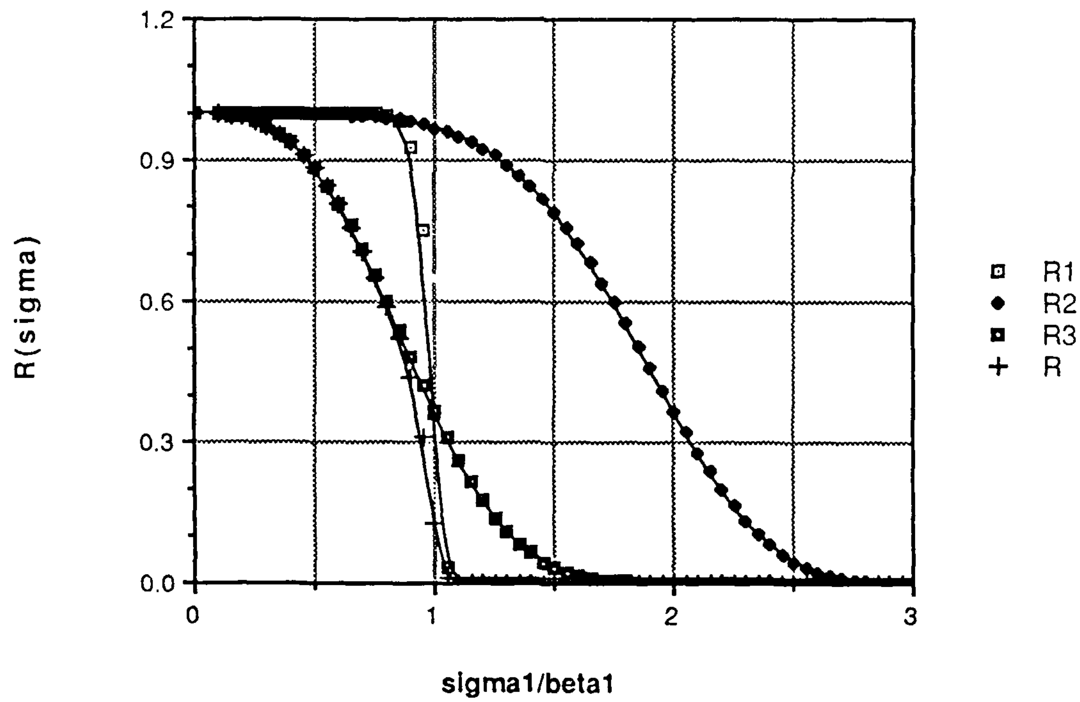

Figure 3-8a : Reliability Vs σ_1/β_1 (C_{12}=0.5, C_{16}=1)

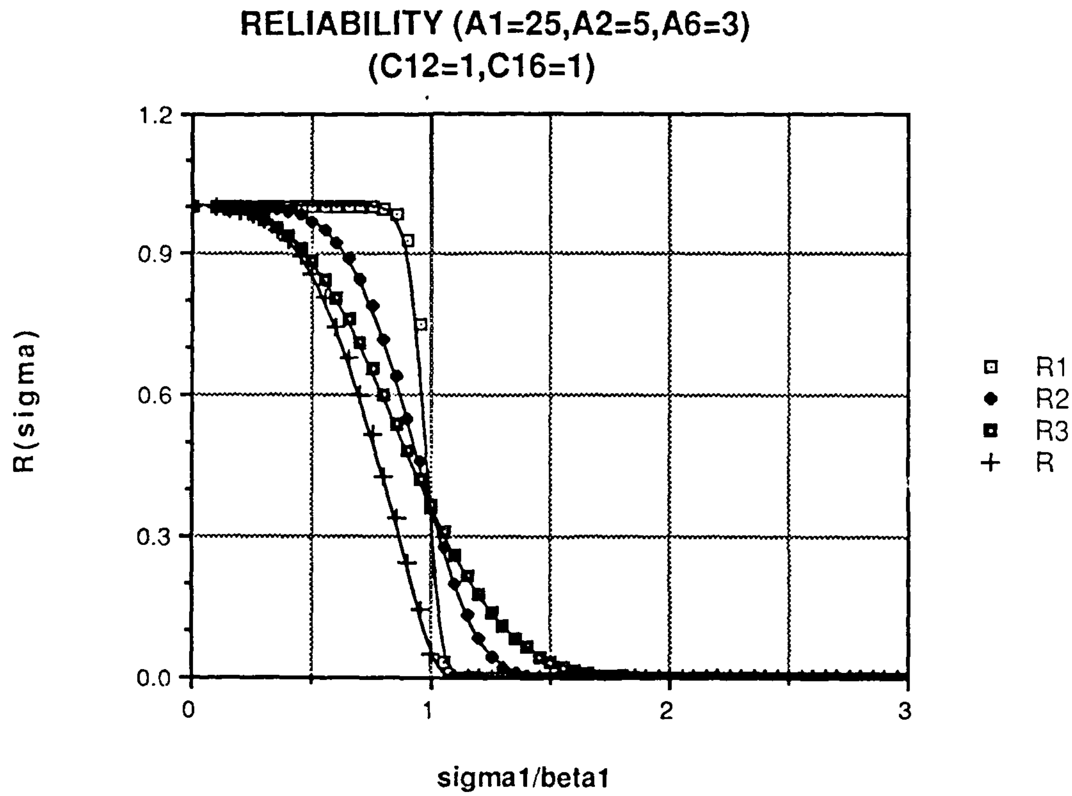

Figure 3-8b : Reliability Vs σ_1/β_1 (C_{12}=C_{16}=1)

If we assume that

$$\beta_1 = V_{12}\,\beta_2,\ \beta_2 = V_{16}\,\beta_6 \tag{3-14}$$

and noting that

$$\sigma_1 = B_{12}\,\sigma_2,\ \sigma_1 = B_{16}\,\sigma_6 \tag{3-15}$$

where

$$B_{12} = \frac{1}{B_{21}},\ B_{16} = \frac{1}{B_{61}},\ V_{12} = \frac{1}{V_{21}},\ V_{16} = \frac{1}{V_{61}} \tag{3-16}$$

then the joint reliability for the independent case can be represented by

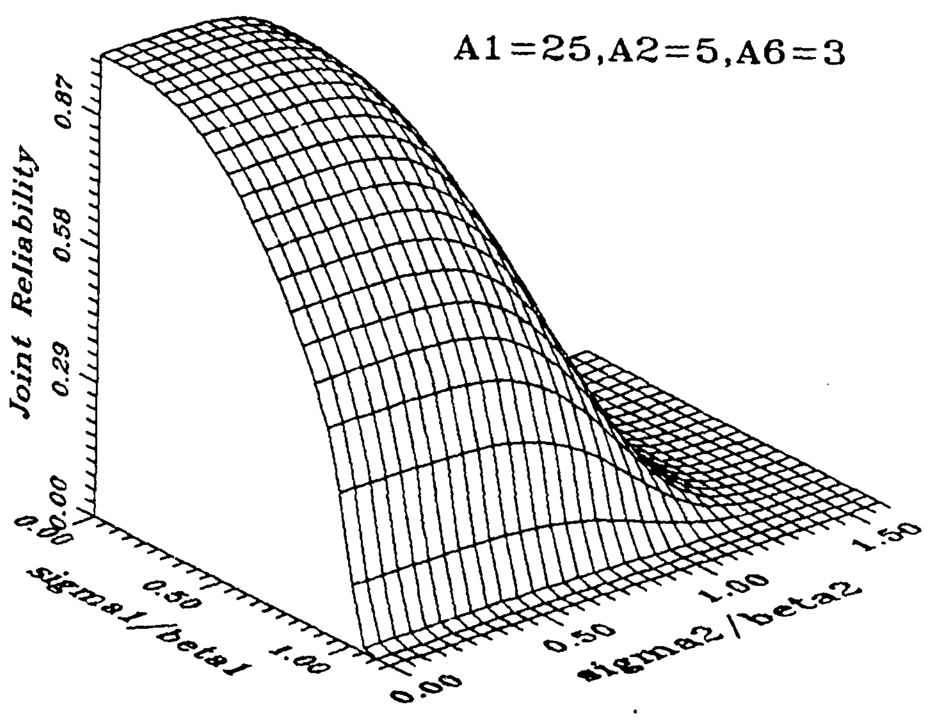

Figure 3-9 : Joint Reliability In 3-D (C_{16}=1)

$$R(\sigma_1,\sigma_2,\sigma_6)=R_1(\sigma_1)R_2(\sigma_2)R_6(\sigma_6)$$
$$=\exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1}\right\}\exp\left\{-\left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\}\exp\left\{-\left(\frac{\sigma_6}{\beta_6}\right)^{\alpha_6}\right\}$$
$$=\exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1}\right\}\exp\left\{-\left(C_{12}\frac{\sigma_1}{\beta_1}\right)^{\alpha_2}\right\}\exp\left\{-\left(C_{16}\frac{\sigma_1}{\beta_1}\right)^{\alpha_6}\right\} \quad (3\text{-}17)$$

where

$$C_{12}=\frac{V_{12}}{B_{12}},C_{16}=\frac{V_{16}}{B_{16}} \quad (3\text{-}18)$$

Furthermore, if C_{12} and C_{16} are given, then C_{26} can be calculated using the chain rule:

$$C_{26}=\frac{V_{26}}{B_{26}}=\frac{\sigma_6}{\sigma_2}\frac{\beta_2}{\beta_6}=\frac{\sigma_1}{\sigma_2}\frac{\sigma_6}{\sigma_1}\frac{\beta_2}{\beta_1}\frac{\beta_1}{\beta_6}=\frac{B_{12}}{V_{12}}\frac{V_{16}}{B_{16}}=\frac{C_{16}}{C_{12}} \quad (3\text{-}19)$$

So, if α_1, α_2, α_6, C_{12}, and C_{16} are given, then the joint reliability function, $R(\sigma_1,\sigma_2,\sigma_6)$, *can be plotted in two dimensional space with respect to* σ_1/β_1. Here we can also introduce the concept of linearized failure CDF to show the effect of the failure CDF at the tail area. To get the linearized failure CDF, denoted by $F^*(\sigma)$, use the same procedure described in Equations (3-5) through (3-9), then

$$F_1^*(\sigma_1)=\alpha_1 \ln\left(\frac{\sigma_1}{\beta_1}\right) \quad (3\text{-}7)$$

$$F_2^*(\sigma_2)=\alpha_2\left\{\ln(C_{12})+\ln\left(\frac{\sigma_1}{\beta_1}\right)\right\} \quad (3\text{-}8)$$

$$F_6^*(\sigma_6)=\alpha_6\left\{\ln(C_{16})+\ln\left(\frac{\sigma_1}{\beta_1}\right)\right\} \quad (3\text{-}20)$$

$$F^*(\sigma_1,\sigma_2,\sigma_6) = \ln\left\{\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} + \left(C_{12}\frac{\sigma_1}{\beta_1}\right)^{\alpha_2} + \left(C_{16}\frac{\sigma_1}{\beta_1}\right)^{\alpha_6}\right\} \quad (3\text{-}21)$$

So it can be noted that the joint linearized failure CDF, $F^*(\sigma_1,\sigma_2,\sigma_6)$ does not show a linear relation in terms of $\ln(\sigma_1/\beta_1)$ except when $\alpha_1=\alpha_2=\alpha_6$, instead it is affected by $F_1^*(\sigma_1)$, $F_2^*(\sigma_2)$, and $F_6^*(\sigma_6)$. That is to say, $F^*(\sigma_1,\sigma_2,\sigma_6)$ is dominated by the weakest value among $F_1^*(\sigma_1)$, $F_2^*(\sigma_2)$, and $F_6^*(\sigma_6)$ so $F^*(\sigma_1,\sigma_2,\sigma_6)$ is always shown on the top of each of graphs as shown in Figures 3-10a and 3-10b. Figures 3-8a through 3-8b show the reliability for different values of C_{12} when α_1, α_2, C_{16} is fixed. These figures are expanded to three dimensional space as shown in Figure 3-9.

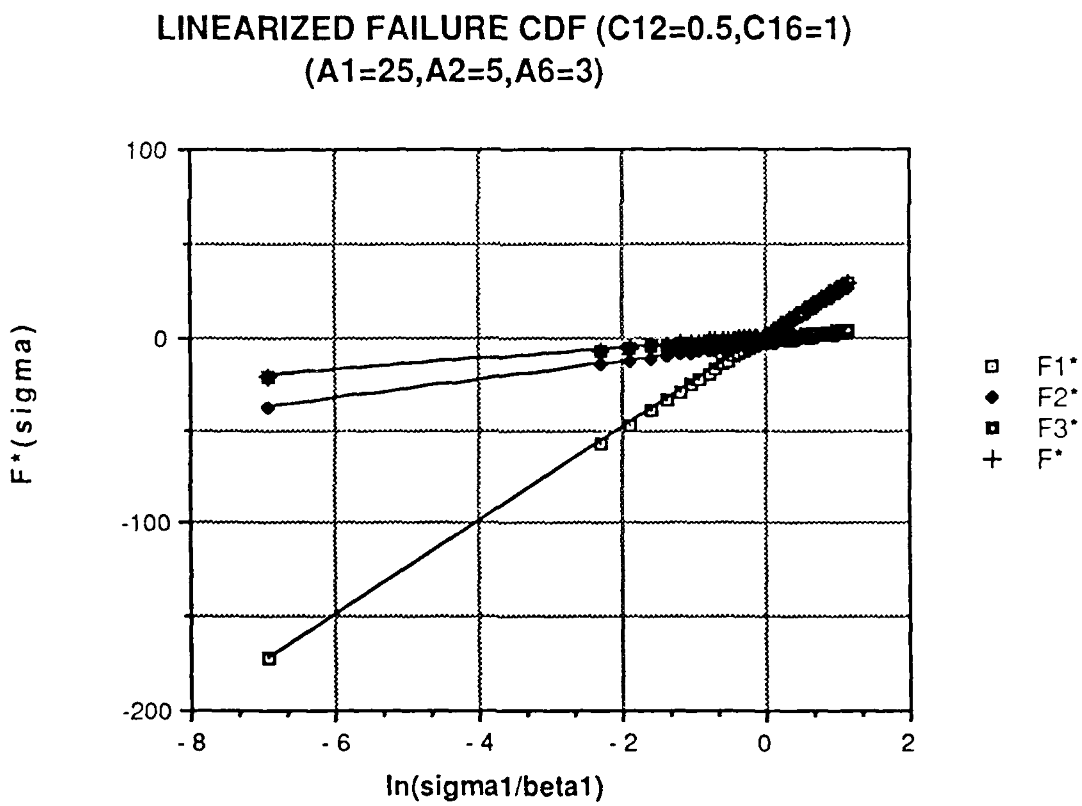

Figure 3-10a : Linearized Failure CDF (C_{12}=0.5, C_{16}=1)

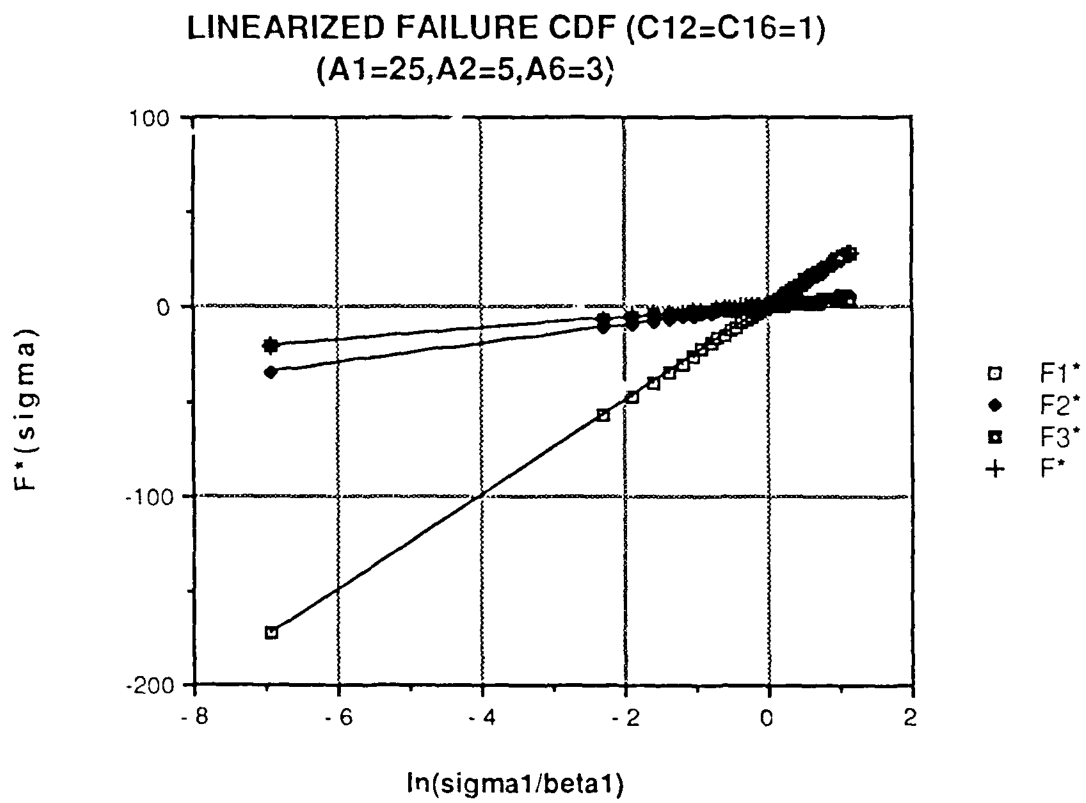

Figure 3-10b : Linearized Failure CDF (C_{12}=C_{16}=1)

As a special case, when $\alpha=\alpha_1=\alpha_2=\alpha_6$, then the joint reliability and linearized failure CDF functions can be simplified by

$$R(\sigma_1, \sigma_2, \sigma_6) = \exp\left\{ -\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha} \left(1 + C_{12}^{\alpha} + C_{16}^{\alpha}\right) \right\} \tag{3-22}$$

$$F_6(\sigma_6) = \alpha\left\{ \ln(c_{16}) + \ln\left(\frac{\sigma_1}{\beta_1}\right) \right\} \tag{3-23}$$

$$F^{*}\sigma_1, \sigma_2, \sigma_6) = \alpha \ln\left(\frac{\sigma_1}{\beta_1}\right) + \ln\left(1 + C_{12}^{\alpha} + C_{16}^{\alpha}\right) \tag{3-24}$$

Observing Equations (3-22) through (3-24), it can be noted that $F_1^{*}(\sigma_1)$, $F_2^{*}(\sigma_2)$, $F_6^{*}(\sigma_6)$, and $F^{*}(\sigma_1,\sigma_2,\sigma_6)$ are parallel each other. And the Joint linearized failure CDF, $F^{*}(\sigma_1,\sigma_2,\sigma_6)$, is also affected by the largest function among F_1^{*}, F_2^{*}, and F_6^{*}. The parameter α is related to the rotation whereas C_{12} and C_{16} are related to the vertical shift of the graph.

IV. PROBABILISTIC FAILURE CONTOUR UNDER COMBINED STRESS

A. PROBABILISTIC FAILURE

For a combined stress failure mechanism, when one stress component does not affect the strength of another component, failure is mechanistically uncoupled, as shown in Figure 4-1. When one stress component affects the strength of another component, failure is then mechanistically coupled, as shown in Figure 4-2.

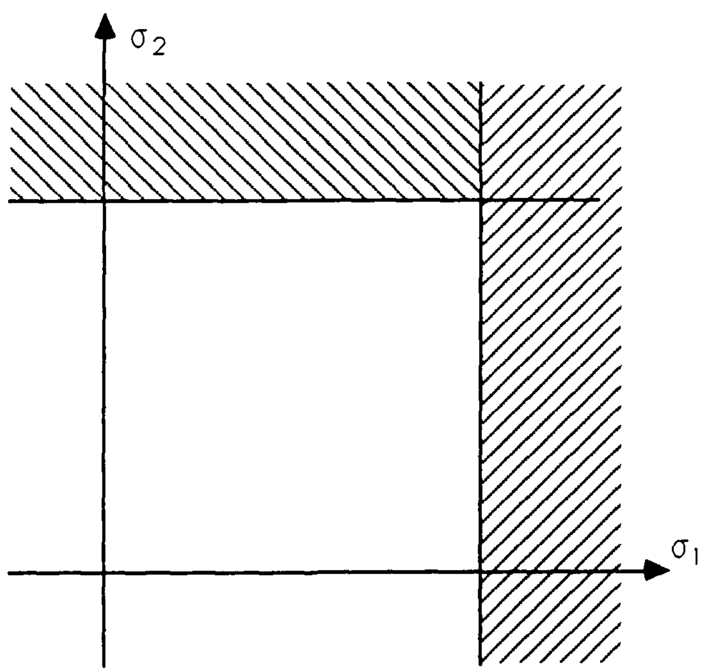

Figure 4-1 : Mechanistically Uncoupled Mechanism

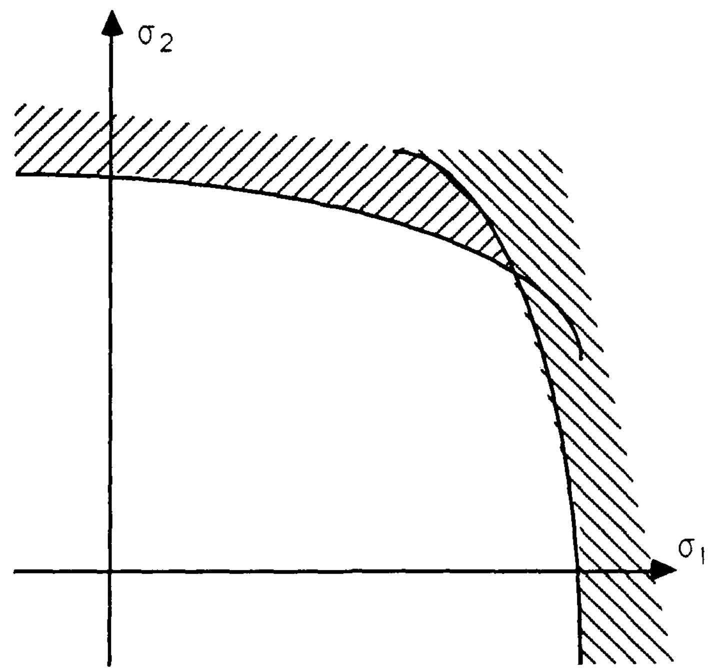

Figure 4-2 : Mechanistically Coupled Mechanism

If the probablistic failure is considered, the mechanistically uncoupled case may contribute to statistical coupling so both failure mechanisms are not interacting and the probablistic failure mechanism is distinguished from the deterministic case. In order words, the condition for the failure mechanism is not fixed, it depends on the stochastic combinations of the intrinsic strengths. So, for the combined stress case, uncoupled independent statistical effects produce phenomenologically coupled statistical contours and the mechanically coupled dependent statistical effects also produce phenomenologically coupled statistical contours. In the following subsection, the joint probabilistic failure

CDF will be discussed for the two and three random variable cases and the effects of the parameters will be shown graphically.

B. Two Random Variable Case

To show the joint failure probability, we here introduce the Weibull distribution function. From Equation (2-62), the joint failure distribution function can be obtained for the two random variable case.

$$F(\sigma_1, \sigma_2) = 1 - \exp\left\{ -\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} - \left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2} \right\} \tag{2-62}$$

From the above equation, (σ_1/β_1) and (σ_2/β_2) can be computed if the values of $F(\sigma_1,\sigma_2)$ and C_{12} are given. Figure 4-3 shows the joint failure CDF in three dimensional space and Figure 4-4 shows corresponding joint failure contour in two dimensional space for the specific value of F. As shown in Figures 4-3 and 4-4, the shapes of the failure function and failure contour depend on the parameter α, which is material dependent. For a small value of α, the failure contour shows a smooth curve but as the value of α increases, the joint failure contour approaches to the shape of a rectangle. This phenomenon is the same for the joint reliability contour as shown in Figure 4-5. It can also be noted that the failure contour is affected by a large value of α between α_1 and α_2. So if anyone of the values of α is larger when compared to another value of α, then the failure contour approaches to the shape of a rectangle, depending on the magnitude of α. These relations are shown in Figures 4-6 and 4-7. Another result, shown in these failure contour graphs, is that the distance between the contour line is small when α is large compared to those when α is small. So the intersection of the function on both axis depends on the value of each α.

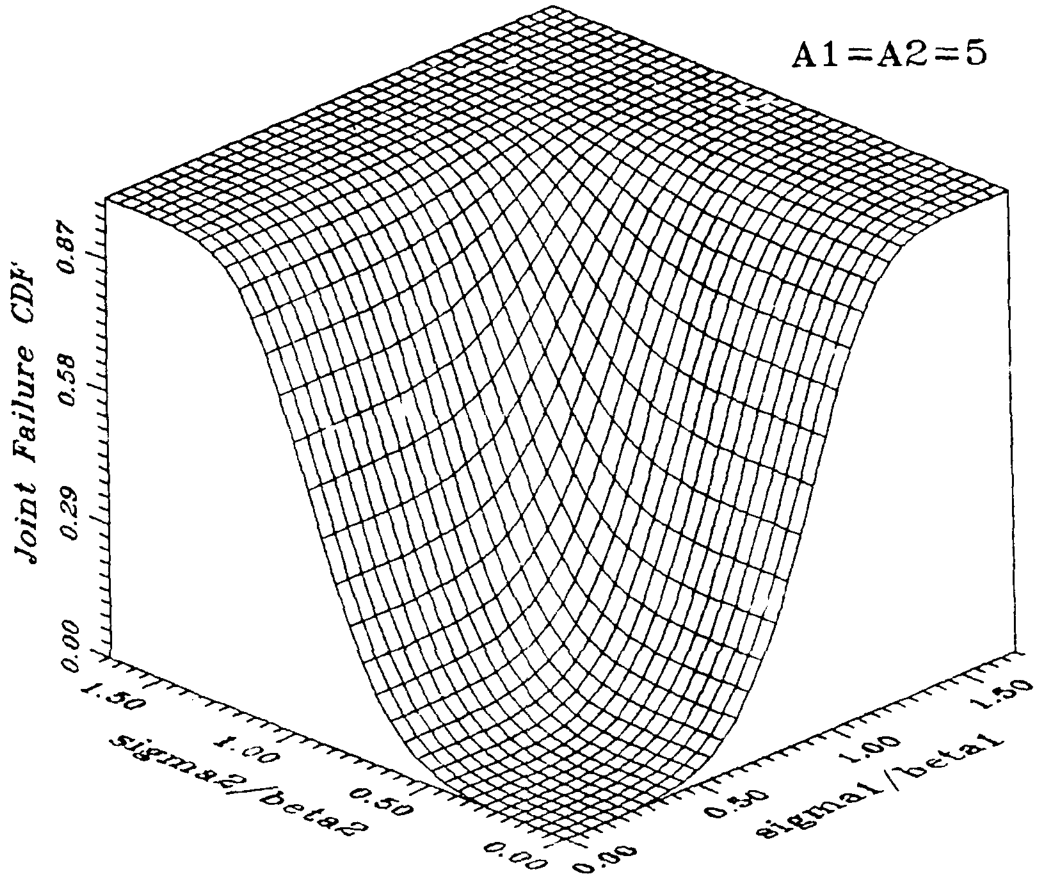

Figure 4-3a : Joint Failure CDF In 3-D (α_1=α_2=5)

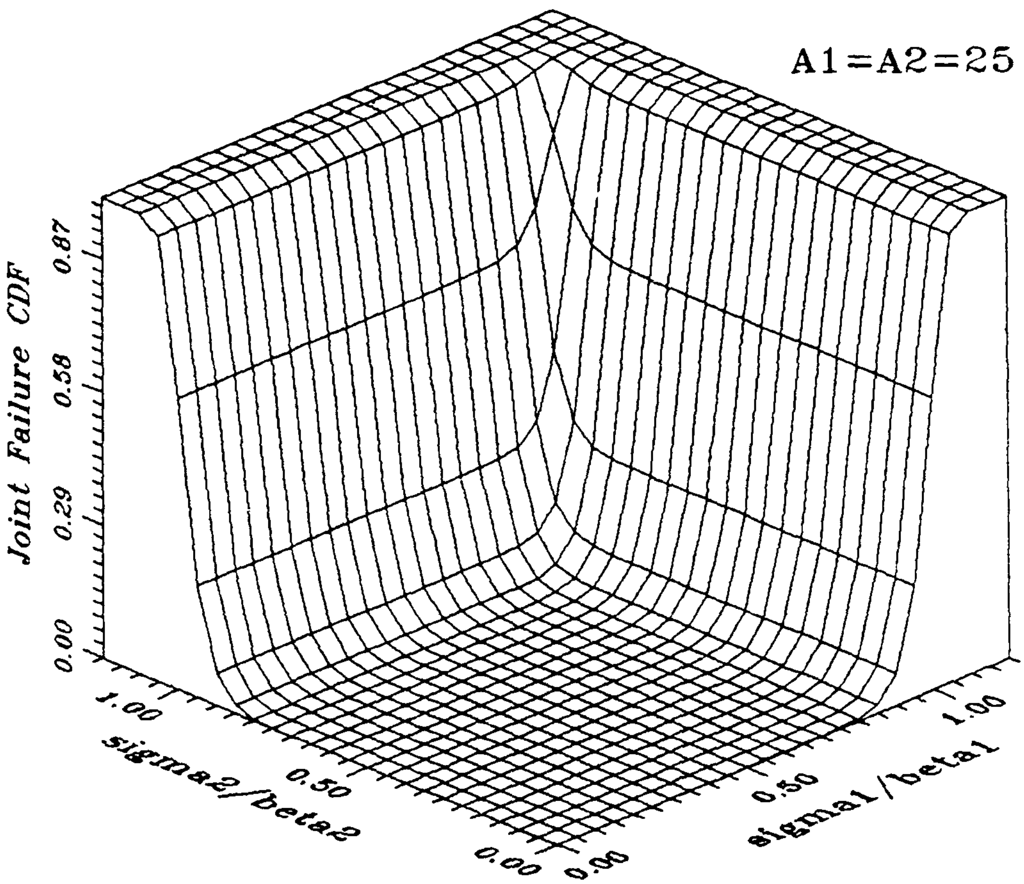

Figure 4-3b : Joint Failure CDF In 3-D (α_1=α_2=25)

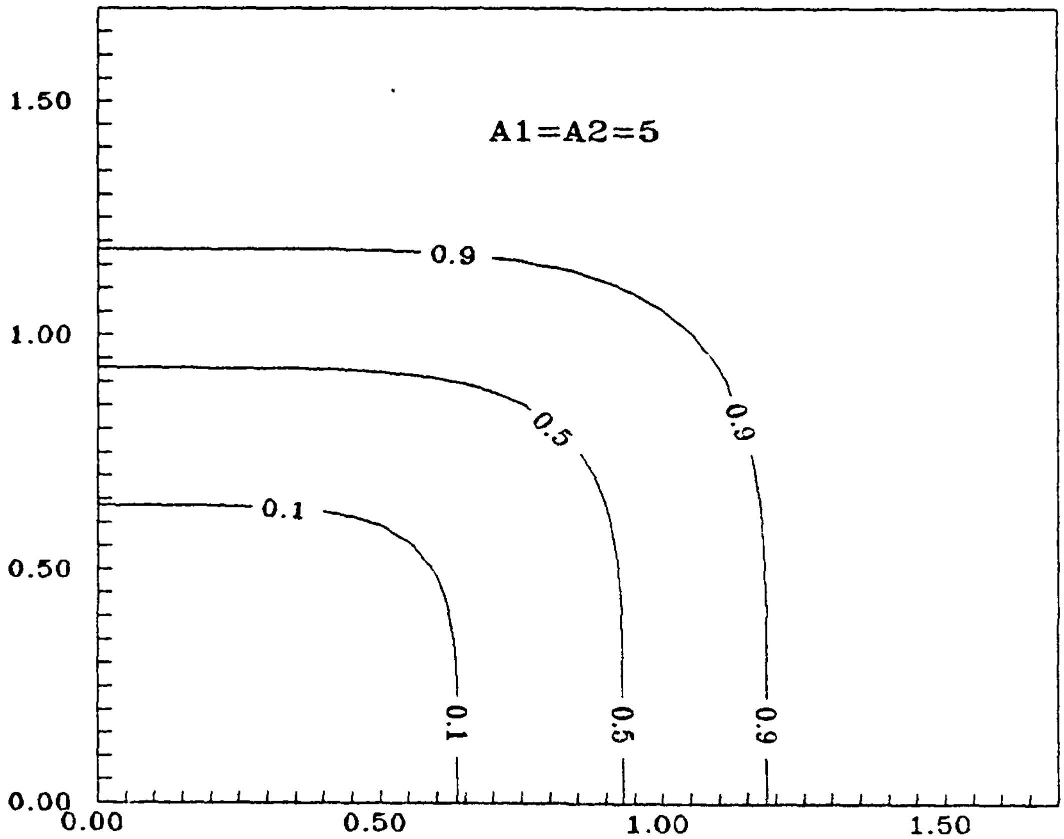

Figure 4-4a : Joint Failure Contour (α_1=α_2=5)

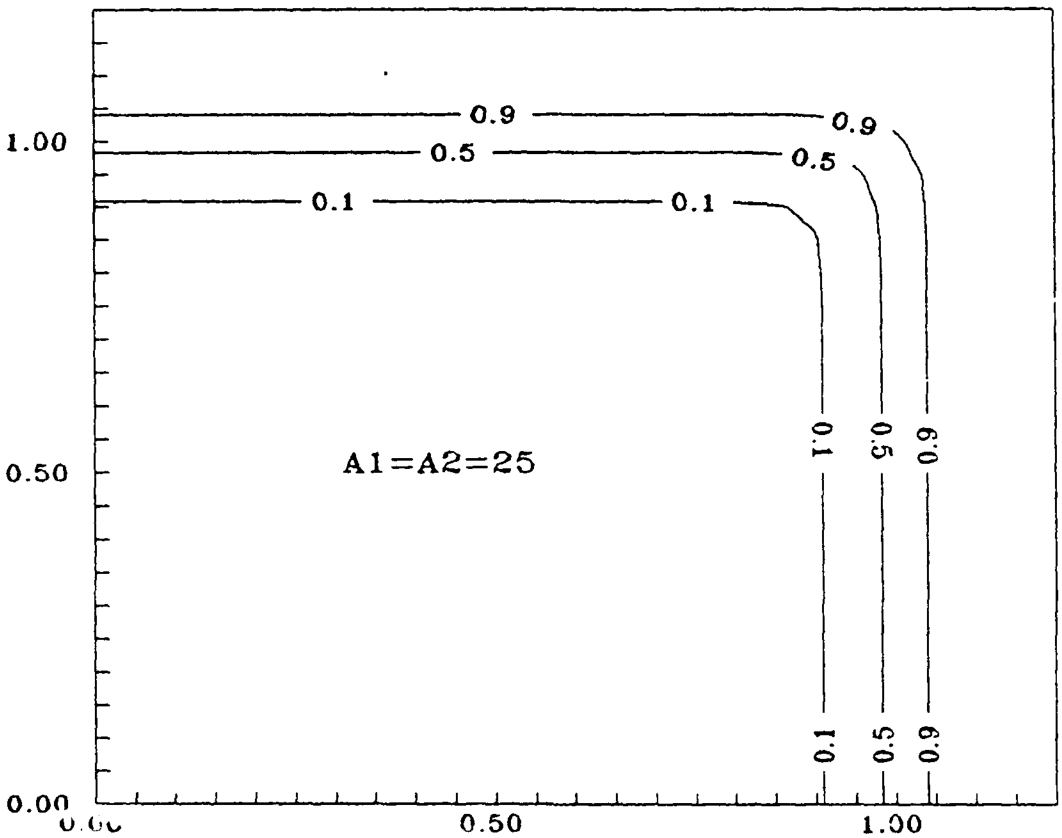

Figure 4-4b : Joint Failure Contour (α_1=α_2=25)

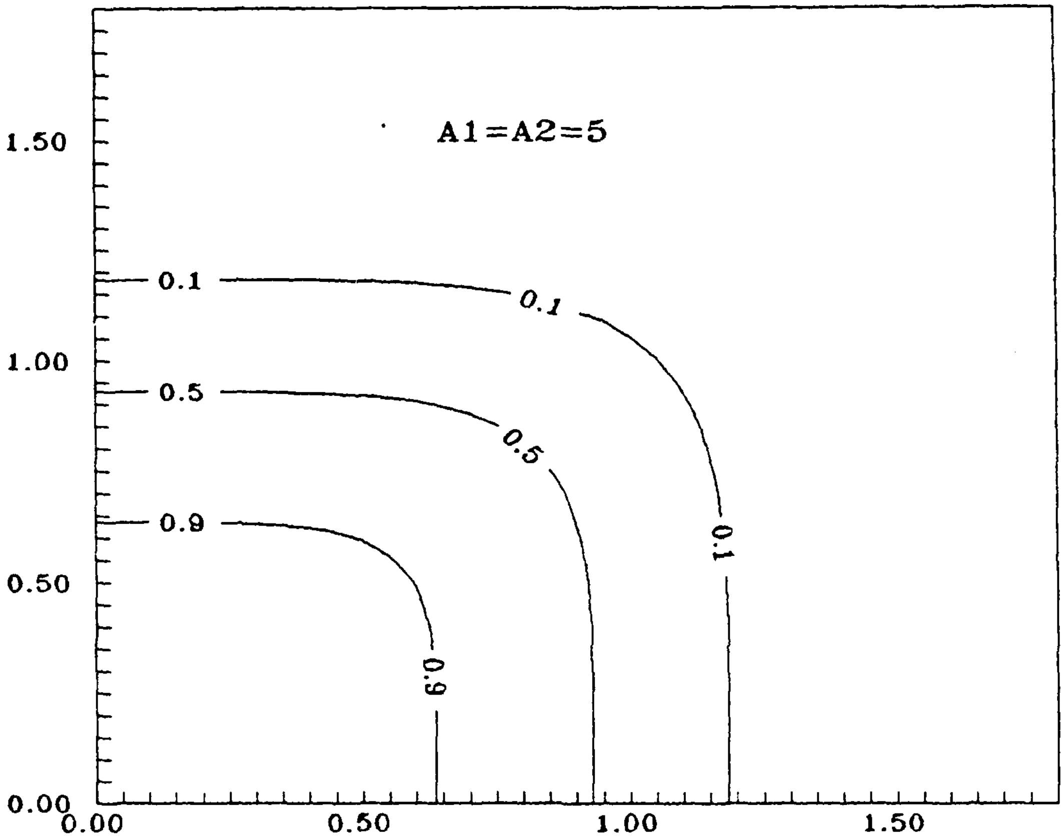

Figure 4-5a : Joint Reliability Contour (α_1=α_2=5)

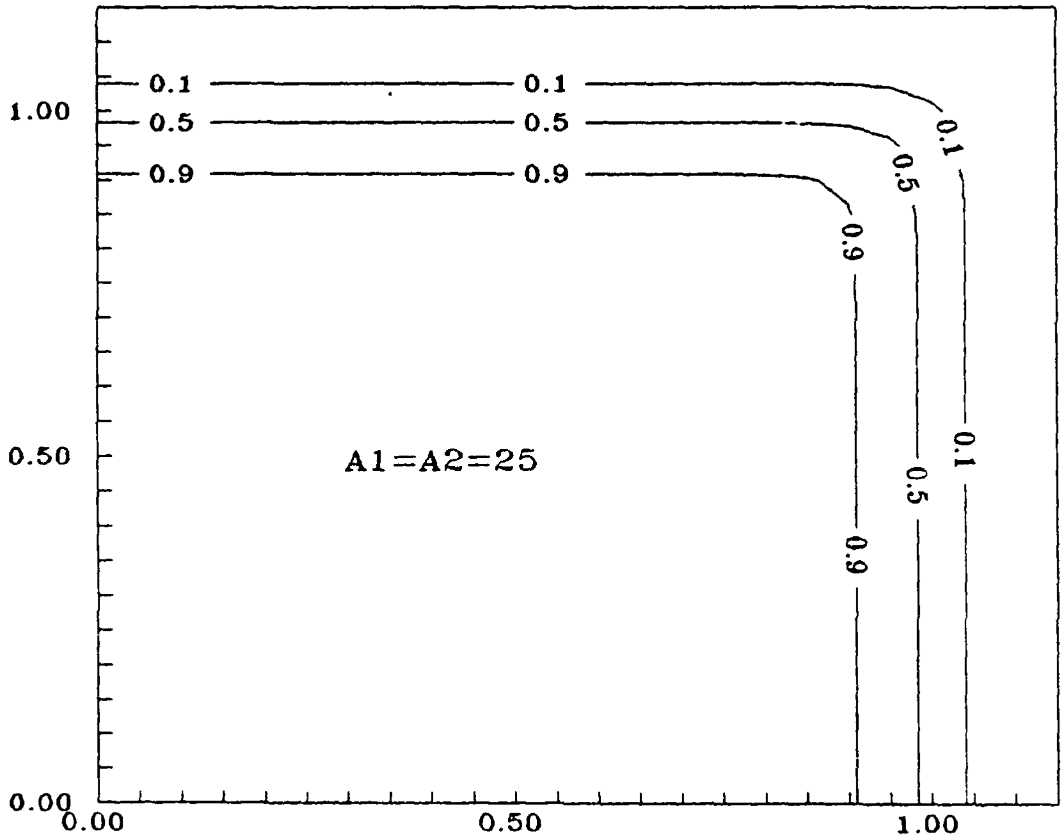

Figure 4-5b : Joint Reliability Contour (α_1=α_2=25)

The contour line of $F(\sigma_1,\sigma_2) = 0.5$ exists around 1.0 in both the normalized x and y axis regardless of α, but as α increases, the F=0.5 contour line approaches to 1.0 in both normalized axis.

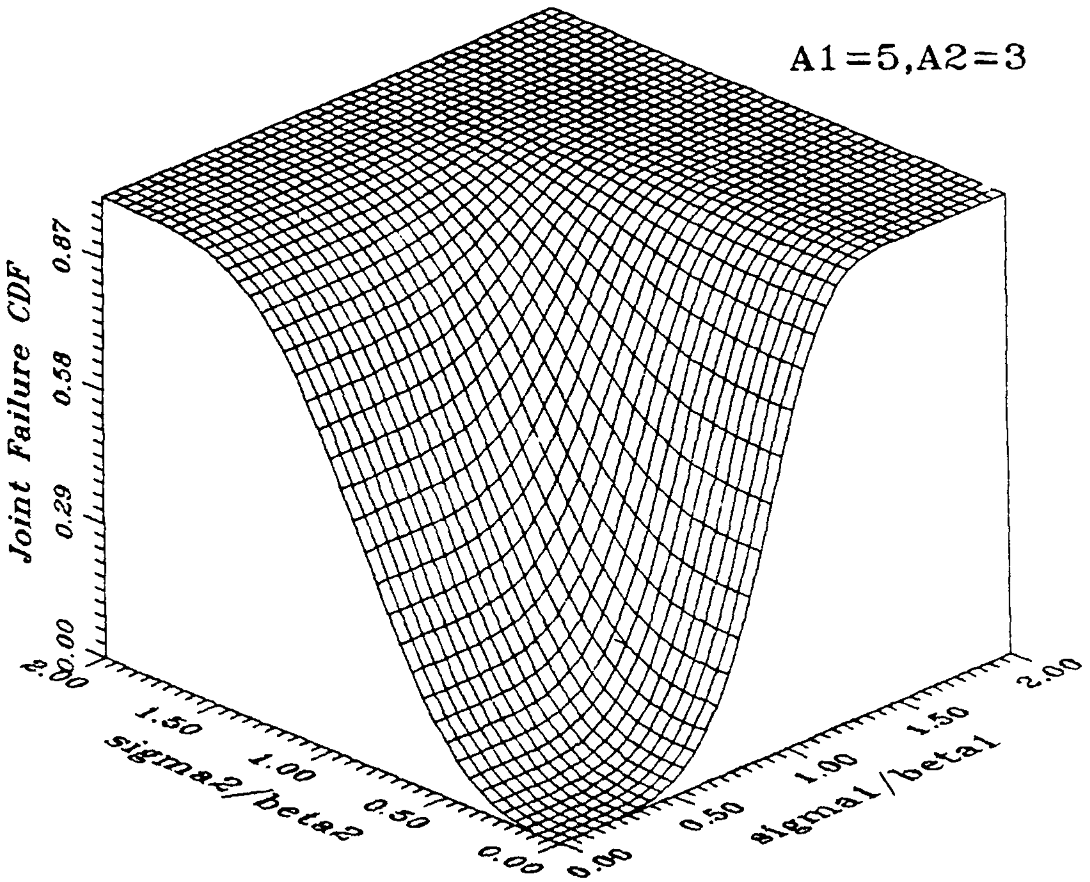

Figure 4-6a : Joint Failure CDF (α_1=5,α_2=3)

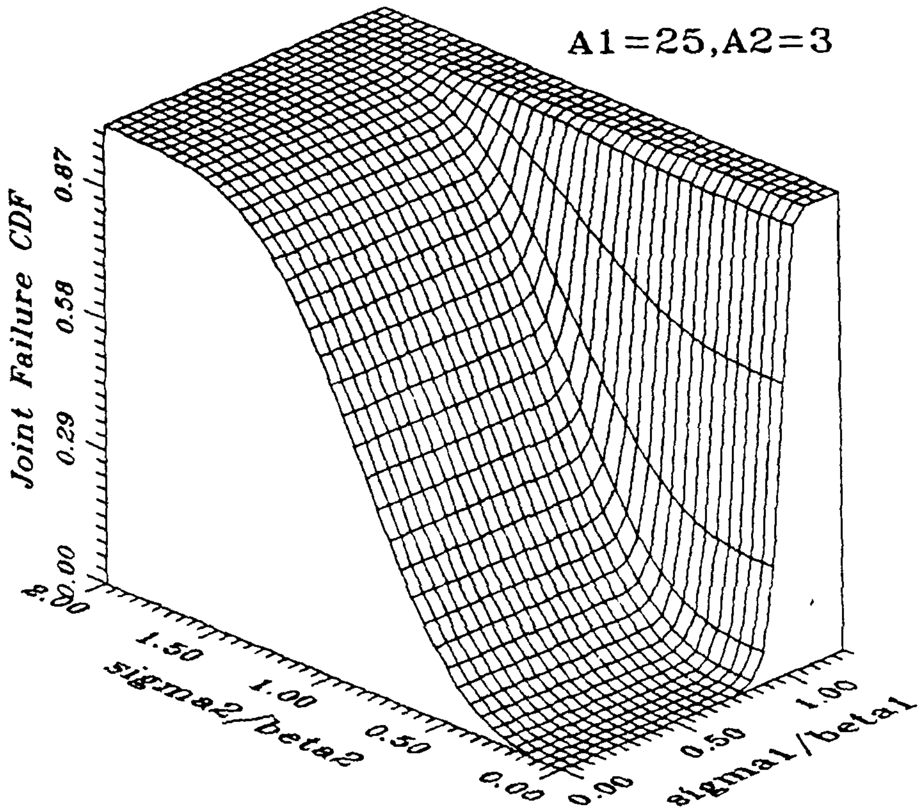

Figure 4-6b : Joint Failure CDF (α_1=25,α_2=3)

As an application, if we compute the stress of a specific part of system such as a rudder or ailerons using the finite element method, then we can estimate the probabilistic failure and reliability of that specific part of the system.

The parameters α and β, which were used in the failure CDF and reliability function, are material dependent so these values should be determined through experimentation and then these parameters can be applied for that specific material.

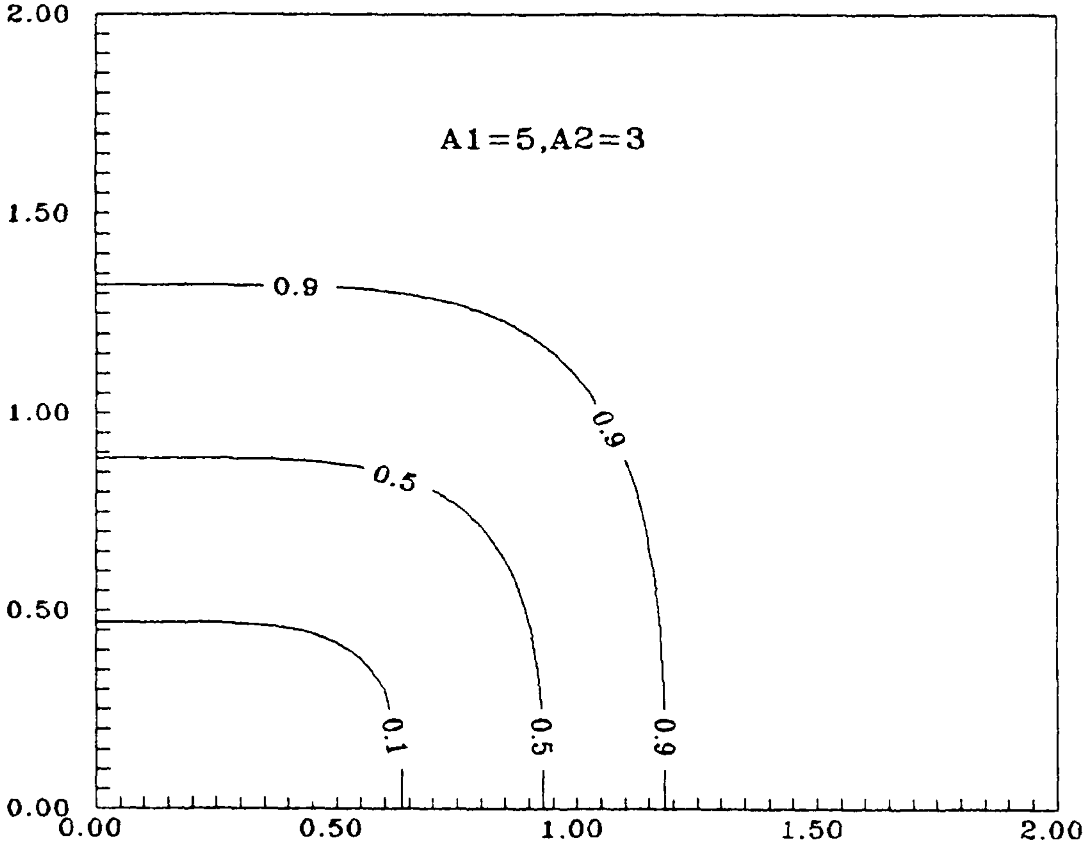

Figure 4-7a : Joint Failure Contour (α_1=5,α_2=3)

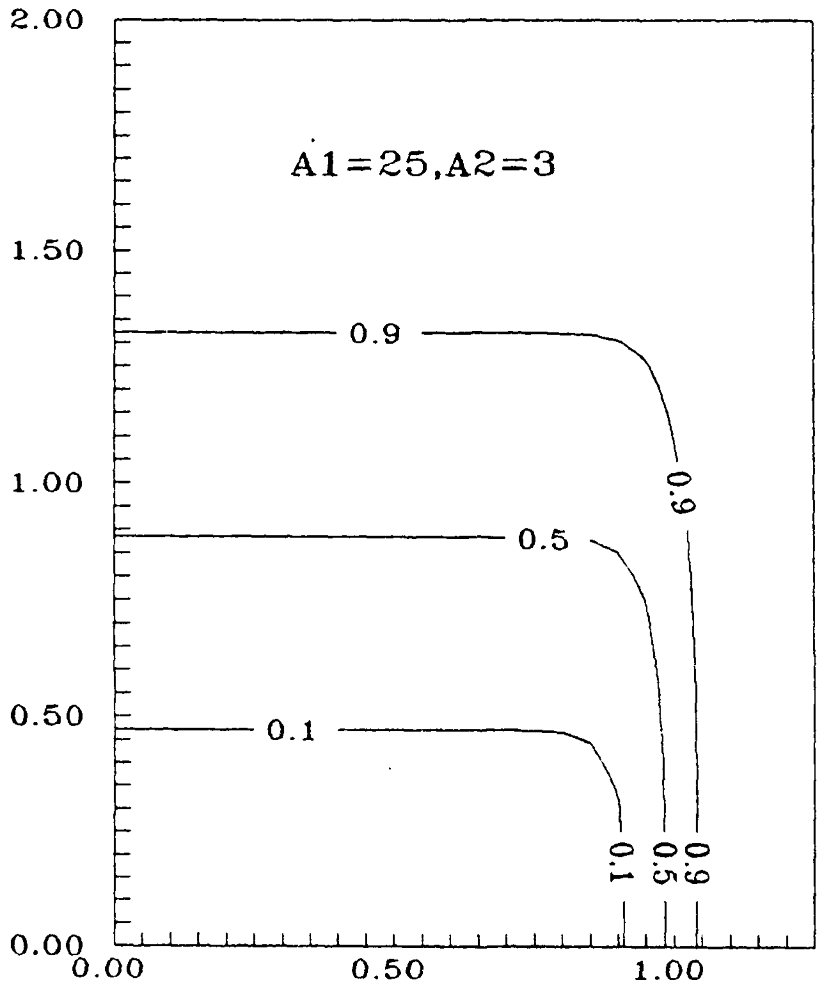

Figure 4-7b : Joint Failure Contour (α_1=25,α_2=3)

C. THREE RANDOM VARIABLE CASE

If the three random variable case is considered, the joint failure CDF can be developed using Equation (2-63).

$$F(\sigma_1, \sigma_2, \sigma_6) = 1 - \exp\left\{ -\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} - \left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2} - \left(\frac{\sigma_6}{\beta_6}\right)^{\alpha_6} \right\}$$

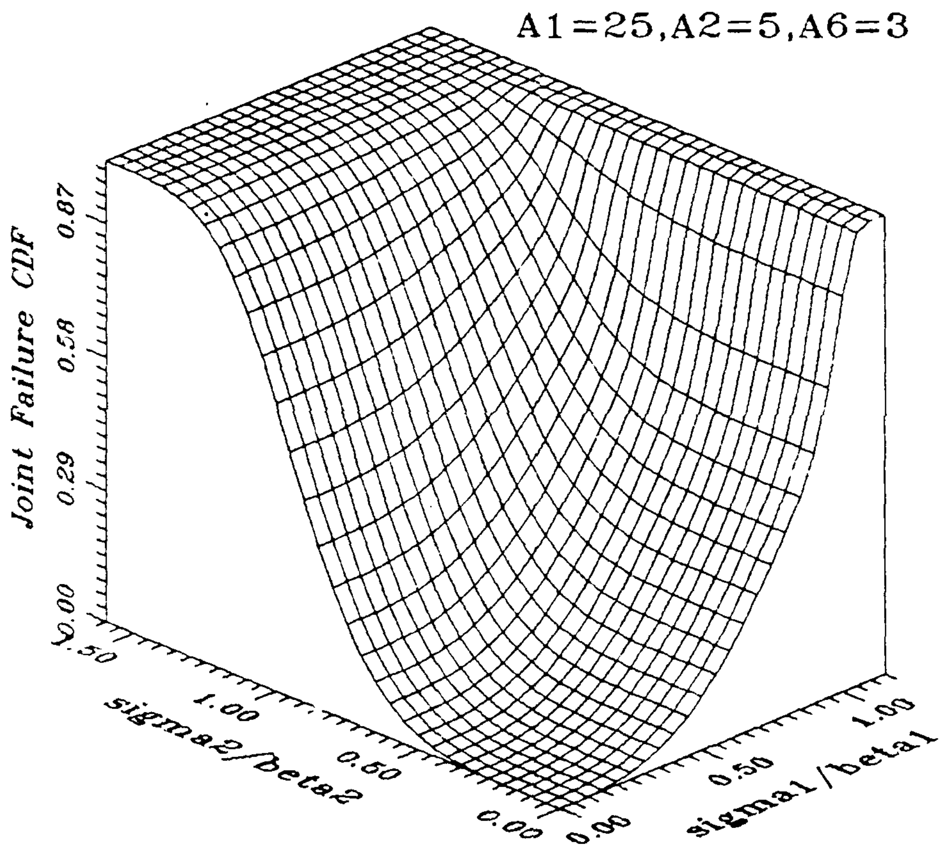

Figure 4-8a . oint Failure CDF (C_{16}=1)

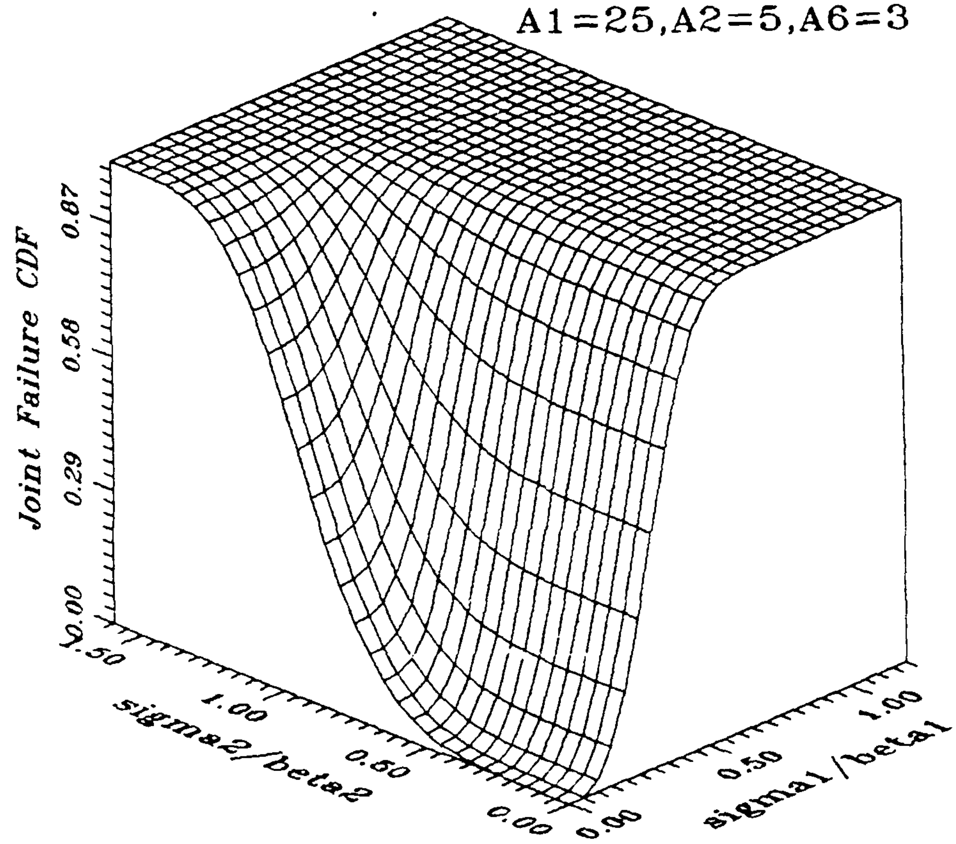

Figure 4-8b : Joint Failure CDF (C_{16}=3)

From the above equation, normalized axis coordinates can be computed if the values of $F(\sigma_1, \sigma_2, \sigma_6)$, C_{12}, and C_{16} are given. And if we specify the value of C_{16}, then we can plot the joint failure CDF in three dimensional space with respect to σ_1/β_1 and σ_2/β_2 as shown in Figure 4-8. As the value of C_{16} increases, the possibility of failure increases, that is to say, the intersection on the normalized x-axis moves to the left because σ_6/β_6 term is embedded in the σ_1/β_1 axis. Figure 4-9 shows the corresponding failure contour line in two dimensional space for the specific value of $F(\sigma_1, \sigma_2, \sigma_6)$.

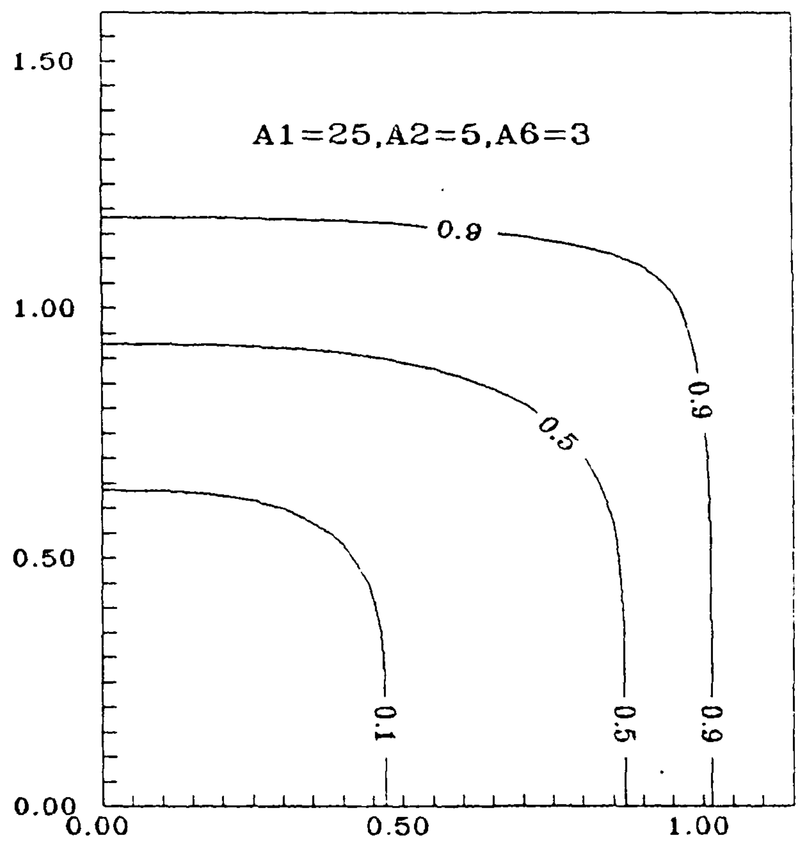

Figure 4-9a : Joint Failure Contour (C_{16}=1)

As proven graphically in the three random variable case, the probability of failure increases as the number of random variables increase. So in a physical sense, a specific composite material will show the highest reliability when the external loads are applied in the pricipal axis direction because there are only two random variable, X_1 and X_2, in the reliability fuction. But when the external loads are applied in the off-principal axis direction, there exist some shear force random variable, X_6, which decreases the reliability of the material.

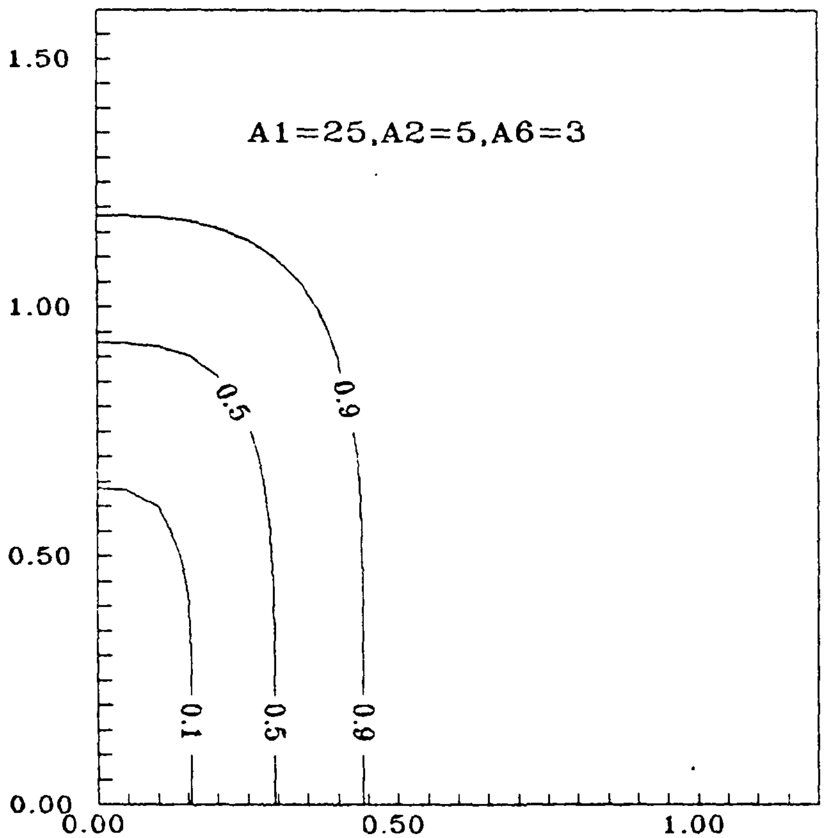

Fig. 4-9b : Joint Failure Contour (C_{12}=3)

As an application, if the stress of the maneuvering aircraft is computed in an arbitrary direction, then the stress can be transformed into fiber, shear, and transverse matrix directions which can then be used to predict the probability of failure at that specific load. Sample calculations were made and the results were analyzed with respect to experimental data in Appendix B.

V. CONCLUSIONS

This study was directed towards deriving the joint reliability and joint failure CDF under a combined stress condition. The results were specialized for the Weibull distribution function based on the observations that the composite material failure is adequately represented by the weakest link model. The effect of the statistical strength parameters, α and β, which are material dependent are illustrated using two and three dimensional graphical representations.

To analyze the reliability of the composite, the inherent statistical strength parameters in the composite's principal direction need to be experimentally measured. Substitution of these parameters in the probabilistic failure criterion will allow for the estimation of the reliability of the composite for any state of stress.

Comparison of joint failure distribution under the restriction of independence to experimental data suggests that mechanistic coupling of the failure mechanism needs to be included in future extensions of the formulation and data of much larger number of samples has to be performed at critical stress ratios to conclusively examine probabilistic independence.

Further studies may be extended to apply these reliability and failure CDF to the specific part of a system which is made of a composite material. To do this, the stress of the specific part of a system such as an elevator or ailerons should be analyzed using such as finite element method. These equations can also be used in a repair problem which requires the least cost and most effective method to analyze the combined stress in a specific part of a system.

So, if we analyze the combined stress, the appropriate fiber and matrix with proper parameters can be selected to make the composite fit to that specific part.

APPENDIX A

Chapter II Equations

1. For the independent case:

$$P(X_1+X_2 : X_1 \le \sigma_1, X_2 \le \sigma_2) = P(X_1 \le \sigma_1) + P(X_2 \le \sigma_2) - P(X_1 \le \sigma_1)\, P(X_2 \le \sigma_2)$$

$$= F_1(\sigma_1) + F_2(\sigma_2) - F_1(\sigma_1)\, F_2(\sigma_2)$$

If the Weibull distribution function is considered, $F_1(\sigma_1)$ and $F_2(\sigma_2)$ can be substituted into the Weibull distribution function:

$$P(X_1 + X_2 : X_1 \le \sigma_1, X_2 \le \sigma_2) = \left[1 - \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1}\right\}\right] + \left[1 - \exp\left\{-\left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\}\right]$$

$$+ \left[1 - \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1}\right\}\right]\left[1 - \exp\left\{-\left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\}\right]$$

By expanding and simplifying

$$P(X_1 + X_2 : X_1 \le \sigma_1, X_2 \le \sigma_2) = 1 - \exp\left\{-\left(\frac{\sigma_1}{\beta_1}\right)^{\alpha_1} - \left(\frac{\sigma_2}{\beta_2}\right)^{\alpha_2}\right\}$$

which is the same result as the Equation (2-32)

2. For the independent case:

$$F(\sigma_1, \sigma_2) = \int_0^{\sigma_1}\left\{\int_{\frac{u}{B_{12}}}^{\infty} f_1(u)\, f_2(v)\,dv\right\}du + \int_0^{\sigma_2}\left\{\int_{B_{12}v}^{\infty} f_1(u) f_2(v)\,du\right\}dv$$

Taking the integrals inside the parenthesis first and substituting limits upon evaluation:

$$F(\sigma_1, \sigma_2) = \int_0^{\sigma_1} f_1(u)\left\{F_2(\infty) - F_2\left(\frac{u}{B_{12}}\right)\right\}du$$

$$+\int_0^{\sigma_2} f_2(v)\{F_1(\infty) - F_1(B_{12}v)\}dv$$

$$F_1(\sigma) = \int_0^{\sigma} f_1(u)du, F_2(\sigma) = \int_0^{\sigma} f_2(u)du$$

Noting $F_1(\infty) = F_2(\infty) = 1$ from the definition of CDF and expanding

$$F(\sigma_1, \sigma_2) = \int_0^{\sigma_1} f_1(u)du - \int_0^{\sigma_1} f_1(u)F_2\left(\frac{u}{B_{12}}\right)du$$

$$+\int_0^{\sigma_2} f_2(v)dv - \int_0^{\sigma_2} f_2(v)F_1(B_{12}v)dv$$

Taking integrals and noting that $F_1(0) = F_2(0) = 0$

$$F(\sigma_1, \sigma_2) = F_1(\sigma_1) + F_2(\sigma_2) - \int_0^{\sigma_1} f_1(u)F_2\left(\frac{u}{B_{12}}\right)du$$

$$-\int_0^{\sigma_2} f_2(v)F_1(B_{12}v)dv$$

3. From Equation (2-33a)

$$F(\sigma_1, \sigma_2) = 2F(\sigma) - 2\int_0^{\sigma} f(u)F(u)du$$

By integration by parts

$$F(\sigma_1, \sigma_2) = 2F(\sigma) - 2\left[\{F(\sigma)\}^2 - \int_0^{\sigma} f(u)F(u)du\right]$$

noting

$$\int_0^{\sigma} f(u)F(u)du = 0.5\{F(\sigma)\}^2$$

then the joint distribution function for independent case can be obtained by

$$F(\sigma_1, \sigma_2) = 2\,F(\sigma) - \{F(\sigma)\}^2$$

4. By expanding and integrating

$$F(\sigma_1,\sigma_2)=\left[1-\exp\left\{-\left(\frac{\sigma_1}{\beta}\right)^{\alpha}\right\}\right]+\left[1-\exp\left\{-\left(\frac{\sigma_2}{\beta}\right)^{\alpha}\right\}\right]$$

$$-\left[-\exp\left\{-\left(\frac{u}{\beta}\right)^{\alpha}\right\}\right]_{o}^{\sigma_1}+\left[-\left(\frac{B_{12}^{\alpha}}{B_{12}^{\alpha}+1}\right)\exp\left\{-\left(\frac{u}{\beta}\right)^{\alpha}-\left(\frac{u}{B_{12}\beta}\right)^{\alpha}\right\}\right]_{o}^{\sigma_1}$$

$$-\left[-\exp\left\{-\left(\frac{v}{\beta}\right)^{\alpha}\right\}\right]_{o}^{\sigma_2}+\left[-\left(\frac{1}{B_{12}^{\alpha}+1}\right)\exp\left\{-\left(\frac{v}{\beta}\right)^{\alpha}-\left(\frac{B_{12}v}{\beta}\right)^{\alpha}\right\}\right]_{o}^{\sigma_2}$$

By substituting and expanding

$$F(\sigma_1,\sigma_2)=\left[1-\exp\left\{-\left(\frac{\sigma_1}{\beta}\right)^{\alpha}\right\}\right]+\left[1-\exp\left\{-\left(\frac{\sigma_2}{\beta}\right)^{\alpha}\right\}\right]$$

$$+\left[\exp\left\{-\left(\frac{\sigma_1}{\beta}\right)^{\alpha}\right\}-1\right]-\left(\frac{B_{12}^{\alpha}}{B_{12}^{\alpha}+1}\right)\left[\exp\left\{-\left(\frac{\sigma_1}{\beta}\right)^{\alpha}-\left(\frac{\sigma_1}{B_{12}\beta}\right)^{\alpha}\right\}-1\right]$$

$$+\left[\exp\left\{-\left(\frac{\sigma_2}{\beta}\right)^{\alpha}\right\}-1\right]-\left(\frac{1}{B_{12}^{\alpha}+1}\right)\left[\exp\left\{-\left(\frac{\sigma_2}{\beta}\right)^{\alpha}-\left(\frac{B_{12}\sigma_2}{\beta}\right)^{\alpha}\right\}-1\right]$$

By rearranging terms and simplifying

$$F(\sigma_1,\sigma_2)=1-\left(\frac{B_{12}^{\alpha}}{B_{12}^{\alpha}+1}\right)\exp\left\{-\left(\frac{\sigma_1}{\beta}\right)^{\alpha}-\left(\frac{\sigma_1}{B_{12}\beta}\right)^{\alpha}\right\}$$

$$-\left(\frac{1}{B_{12}^{\alpha}+1}\right)\exp\left\{-\left(\frac{\sigma_2}{\beta}\right)^{\alpha}-\left(\frac{B_{12}\sigma_2}{\beta}\right)^{\alpha}\right\}$$

noting $\sigma_1 = B_{12}\sigma_2$, then the joint distribution function of the Weibull model under bi-axial stress σ_1 and σ_2 can be simplified by

$$F(\sigma_1, \sigma_2) = 1 - \exp\left\{-\left(\frac{\sigma_1}{\beta}\right)^{\alpha} - \left(\frac{\sigma_2}{\beta}\right)^{\alpha}\right\}$$

5. From Equation (2-50)

$$R(\sigma_1, \sigma_2) = \int_{\sigma_2}^{\infty}\int_{\sigma_1}^{\infty} f_1(u) f_2(v) du dv$$

$$= \int_{\sigma_2}^{\infty} \left\{F_1(\infty) - F_1(\sigma_1)\right\} f_2(v) dv$$

noting $F_1(\infty) = F_2(\infty) = 1$

$$R(\sigma_1, \sigma_2) = \left\{1 - F_1(\sigma)\right\} \int_{o}^{\infty} f_2(v) dv$$

integrating then we finally got the Equation (2-57):

$$R(\sigma_1, \sigma_2) = \{1 - F_1(\sigma)\}\{1 - F_2(\sigma)\}$$

APPENDIX B

Structural Applications

In many structural applications for aircraft, the composite laminae are oriented at an angle to the tensile load. Take for example the skin on a wing, where the filament angle is oriented primarily along the direction of tension due to bending. Transforming the stress by an angle θ to an off-axis introduces a shear coupling that will compensate for the bending/rotational coupling of a swept wing. Such a material loading relation is representative of that indicated in Figure B-1. We will explore the combined stress state for such a configuration and the structural reliability as affected by the off-axis angle θ. Since we are addressing a spatially homogeneous state of stress, both stress analysis and the substitution of the combined stress in the probabilistic failure criterion can be carried out explicitly.

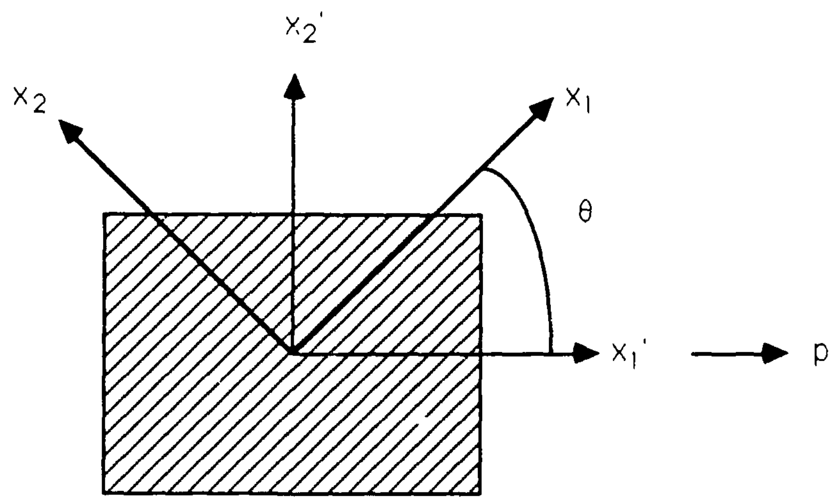

Figure B-1 : Transformation Of Coordinates

For the stress analysis, a load p is applied to the composite material and we denote the direction of p to be x_1', then p can be transformed to the principal direction of the composite; x_1 and x_2 where

x_1 : fiber direction

x_2 : transverse matrix direction

and the angle between x_1' and x_1-axis can be defined as θ which is positive clockwise. Using the tensor relation, the stress can be transformed from x_1'-x_2' axis to x_1-x_2 axis as follows:

$$\sigma_{ij} = a_{mi}\, a_{nj}\, \sigma_{mn}' \tag{B-1}$$

where

i, j, m, n = 1, 2 for 2-dimension

$\sigma_{11} = p$

$\sigma_{22}' = \sigma_{12}' = \sigma_{21}' = 0$

$$a_{ij} = \begin{pmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{B-2}$$

Expanding Equation (B-1) for $\sigma_{11}, \sigma_{12}, \sigma_{22}$, then

$$\begin{aligned} \sigma_{11} &= a_{m1}\, a_{n1}\, \sigma_{mn}' \\ &= a_{11}\, a_{11}\, \sigma_{11}' + a_{11}\, a_{21}\, \sigma_{12}' + a_{21}\, a_{12}\, \sigma_{21}' + a_{21}\, a_{21}\, \sigma_{22}' \\ &= a_{21}^2\, \sigma_{11}' \end{aligned}$$

$$\sigma_{11} = p \cos2(\theta) \tag{B–3}$$

For the same reason

$$\sigma_{12} = -p \cos(\theta) \sin(\theta) \tag{B–4}$$

$$\sigma_{22} = p \sin(\theta) \tag{B–5}$$

To calculate the reliability associated with the applied load p, the stress components in terms of p (Equations (B-3), (B-4), and (B-5)) can be substituted into Equation (2-83) to find the joint failure CDF for the specified external load p.

We note that Equation (2-83) is specialized for the case that the effect of σ_{11}, σ_{12}, and σ_{22} on the respective failure modes are independent. Substitution yields:

$$F(p,\theta) = 1 - \exp\left\{ -\left(\frac{p\cos^2(\theta)}{\beta_1}\right)^{\alpha_1} - \left(\frac{p\sin^2(\theta)}{\beta_2}\right)^{\alpha_2} - \left(\frac{p\cos(\theta)\sin(\theta)}{\beta_6}\right)^{\alpha_6} \right\}$$

(B-6)

where

$\sigma_1 = \sigma_{11}, \sigma_2 = \sigma_{22}, \sigma_6 = \sigma_{12}$

Comparing Equation (B-6) to (2-83) and (3-18), it can be noted that

$B_{12} = \cot^2(\theta), B_{16} = -\cot(\theta)$

$C_{12} = V_{12}\tan^2(\theta), C_{16} = -V_{16}\tan(\theta)$

This joint failure CDF in terms of p and θ can be plotted in terms of p for an specified θ. Similarly, from the Equation (B-6), the linearized failure CDF can be obtained by

$$F^{*}(p,\theta) = \ln\left[\left\{\frac{p\cos^2(\theta)}{\beta_1}\right\}^{\alpha_1} + \left\{\frac{p\sin^2(\theta)}{\beta_2}\right\}^{\alpha_2} + \left\{\frac{p\cos(\theta)\sin(\theta)}{\beta_6}\right\}^{\alpha_6} \right]$$

(B-7)

In order to investigate the dependency of the failure mechanism, experimental data from Sun and Yamada [Ref. 4] is examined. From the experiment, the following parameters were obtained for fiber, transverse, and shear force using uni-axial tests.

$\alpha_1 = 20.5, \beta_1 = 127000$

$\alpha_2 = 5.66, \beta_2 = 1070$

$\alpha_6 = 8.96, \beta_6 = 3078$

Substituting these strength parameters into Equations (B-6) and (B-7), the probability of failure for load p applied to different angles can be examined. Figures (B-2a) through (B-6b) compare the joint failure CDF to the experimental data for different angles. These data will be examined individually for each of the five angles where experimental data is available.

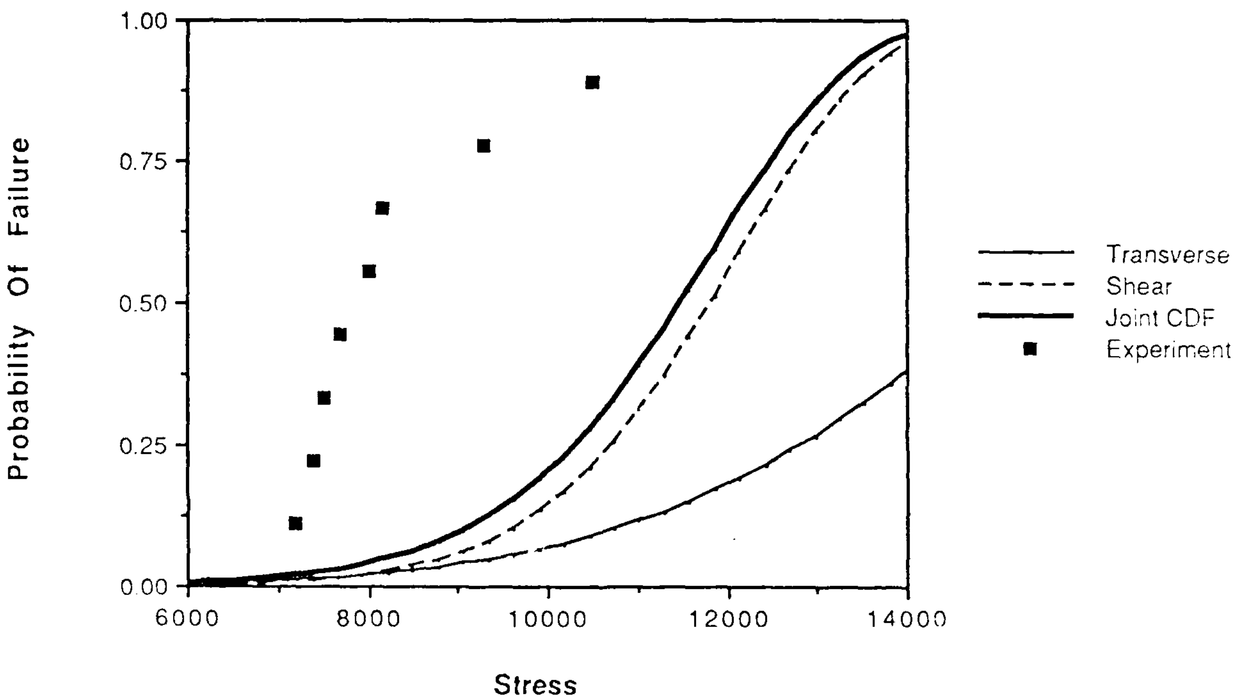

Figure B-2a : Comparison Of Experimental Data with Joint Failure CDF For Independent Case (15 Deg.)

At $\theta = 15°$, even when the load is closely aligned with the fiber angle, Figure B-2a shows that the fiber strength statistics has no effect on the combined failure probability by noting that the fiber failure curve is off-scale for the high p region. That is, the failure is always dominated by a combination of matrix transfer strength and the matrix shear strength. The interaction of transverse and shear can be better observed in the linearized Weibull plot, Figure B-2b, from which it can be seen that in the lower tail (low p region) the transverse strength dominates and in the upper tail (high p region) the shear strength dominates.

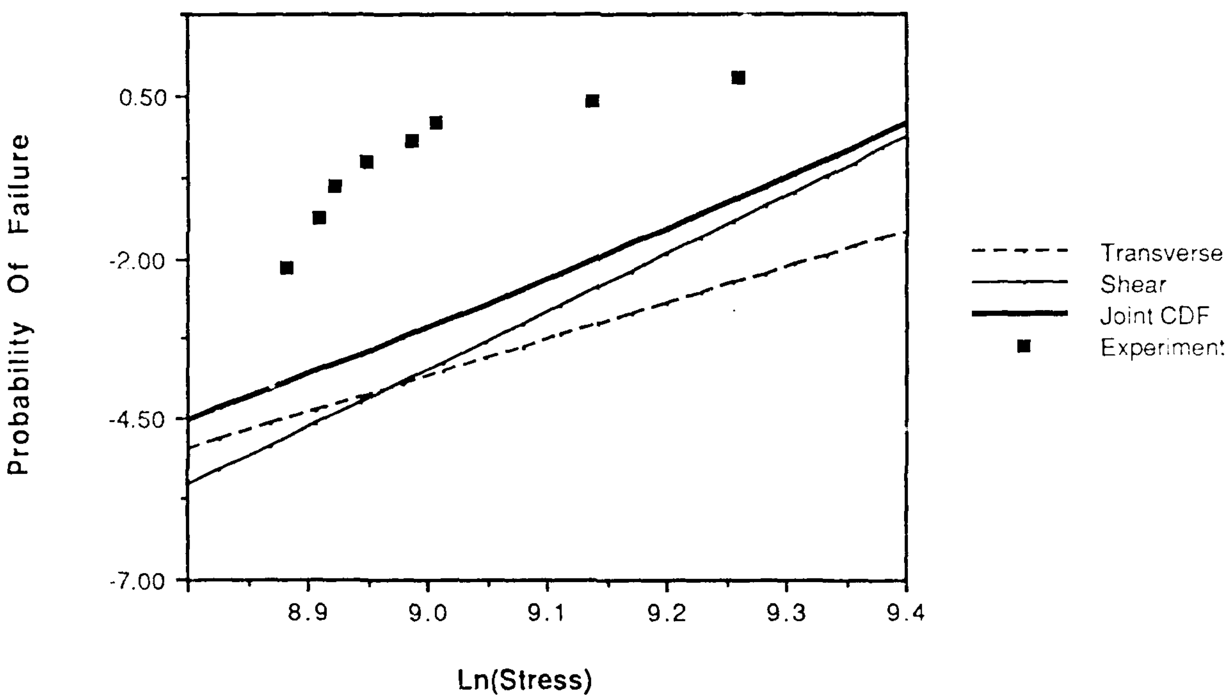

Figure B-2b : Comparison Of Experimental Data With Joint Failure CDF For Independent Case (Linearized, 15 Deg.)

While the experimental data are not close to the predicted joint failure CDF under the simplification of independence, its significance can not be evaluated because of the small sample size. With the eight samples in the current case, neither the upper tail nor the lower tail can be observed,thus the failure probability coupling remains unresolved. The location discrepancy perhaps has more significance. It suggests that combined transverse and shear stresses have weakened the composites and that the failure mechanistic coupling needs to be treated.

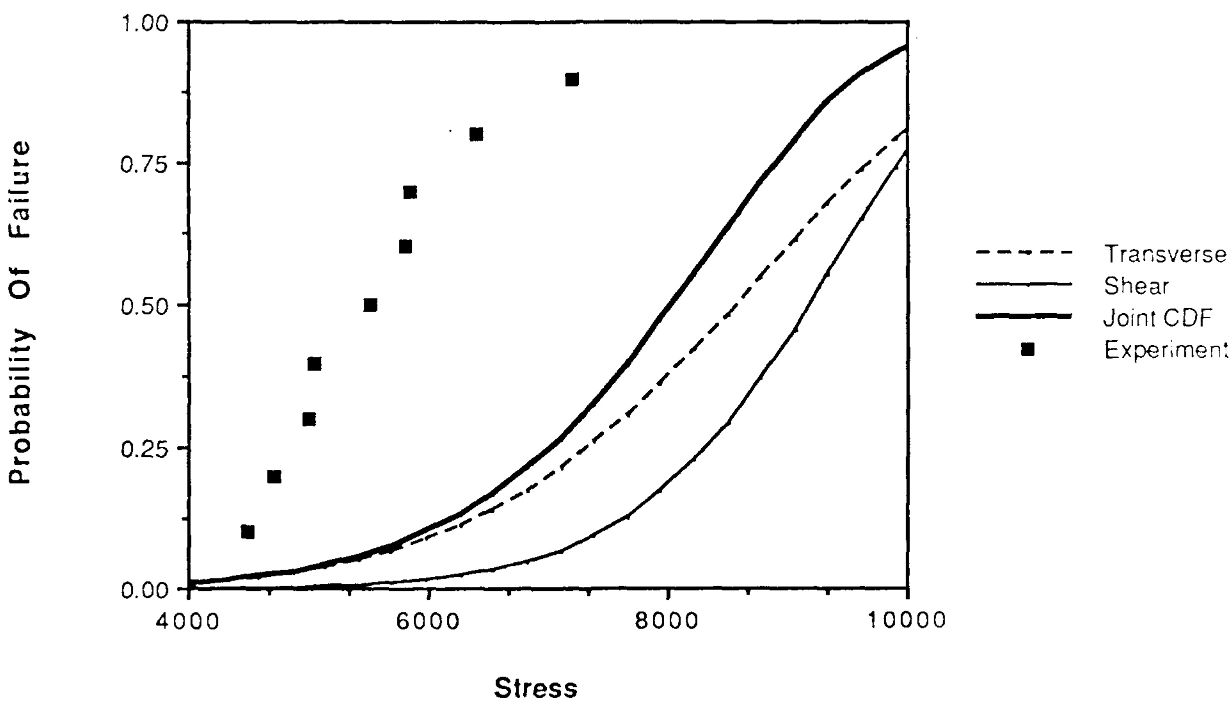

Figure B-3a : Comparison Of Experimental Data With Joint Failure CDF For Independent Case (20 Deg.)

At θ = 20°, Figures B-3a and 3b show that even with a slight angle change of 5°, the effect of the shear component diminishes rapidly. The transition region has shifted higher on the upper tail. Physically, it means that unless a large number of samples are tested, only the strongest of the samples will fail in shear, and the remaining will fail by transverse stress. For this reason, and again because of the small number of samples, the shape indicated by the experimental data has no significance. The location difference between the data and prediction suggests that mechanistic coupling needs to be included.

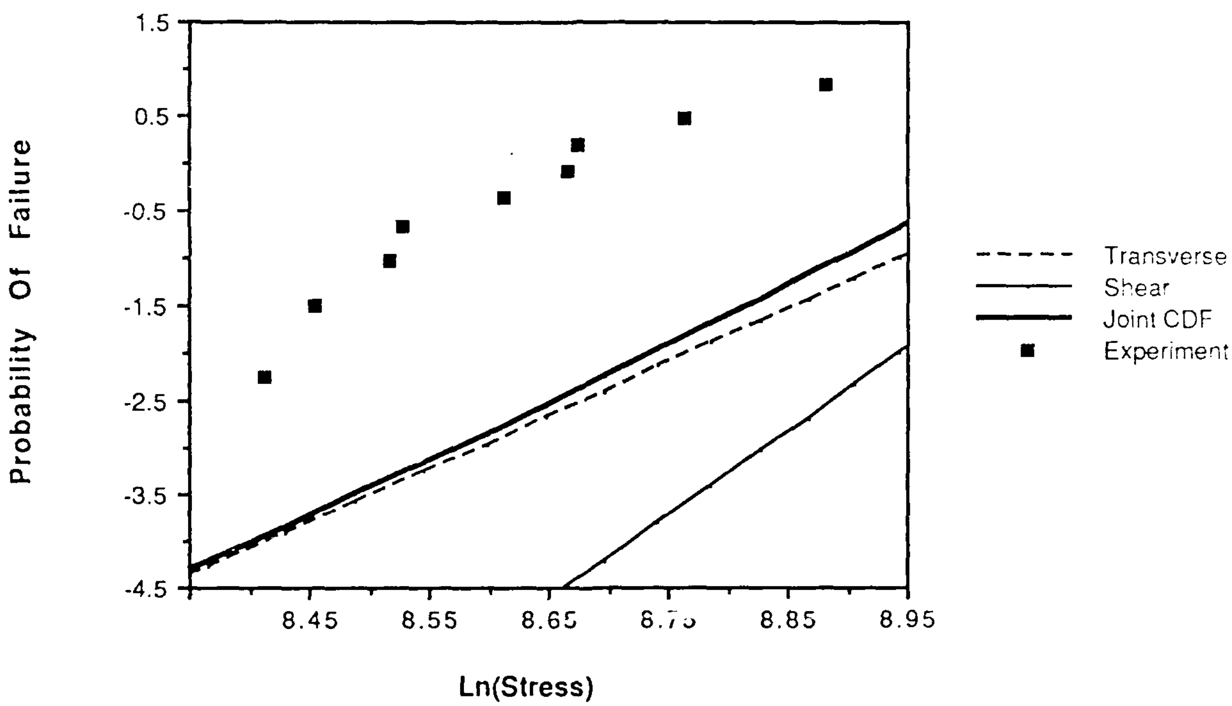

Figure 3-3b : Comparison Of Experimental Data WithJoint Failure CDF For Independent Case (Linearized, 20 Deg)

At $\theta = 30°$, Figures B-4a and 4b show that the effect of shear is shifted to even a higher upper tail. All comments on $\theta = 20°$ data apply to $\theta = 30°$ as well.

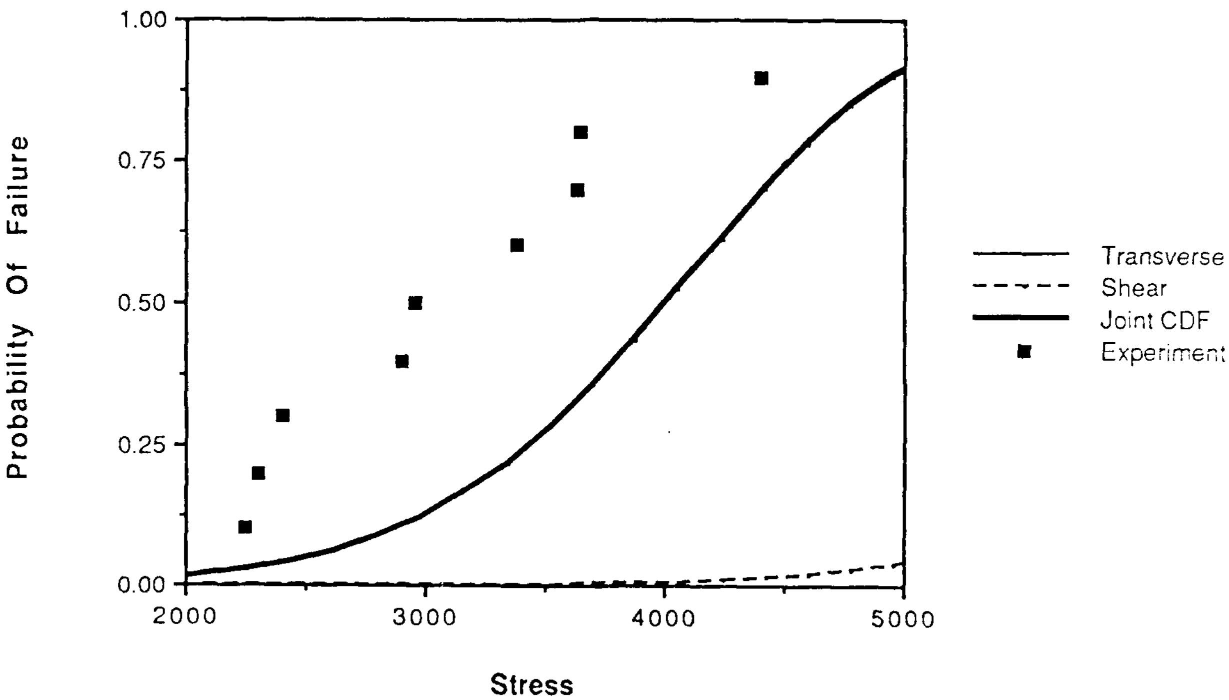

Figure B-4a : Comparison Of Experimental Data With Joint Failure CDF For Independent Case (30 Deg.)

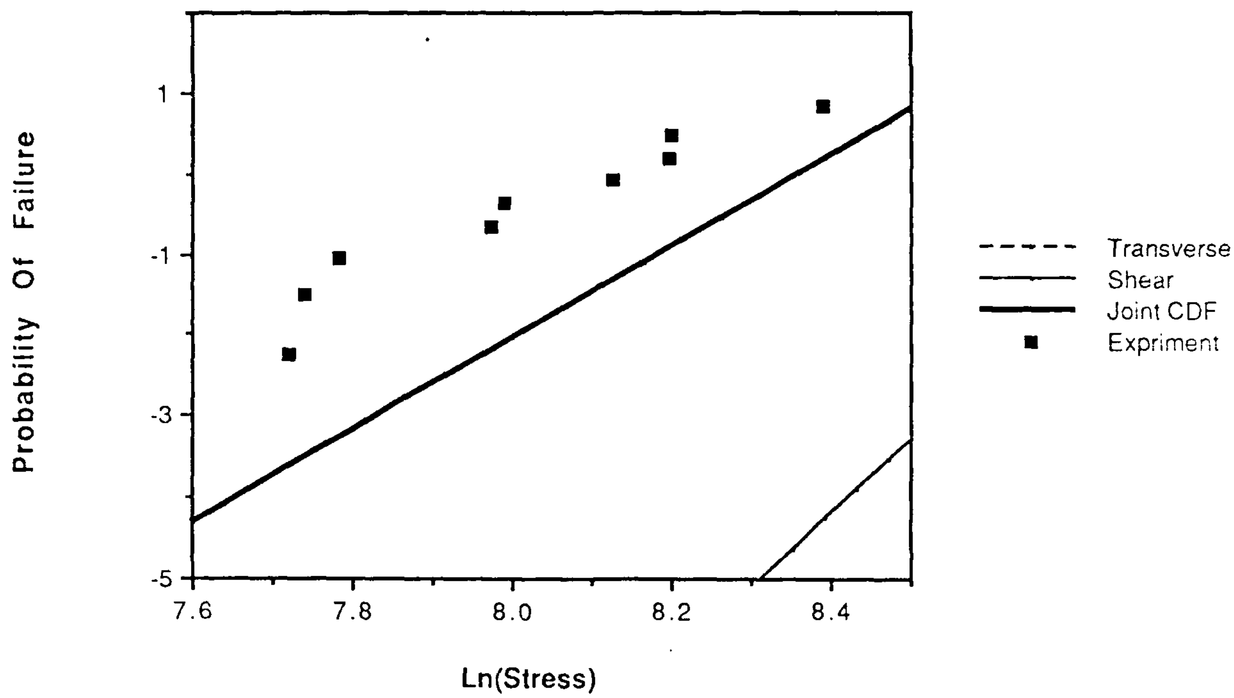

Figure B-4b : Comparison Of Experimental Data With Joint Failure CDF For Independent Case (Linearized, 30 Deg.)

At θ = 45°,Figures B-5a and 5b show that the effect of shear is practically nonexistent; in fact, it can not be observed from the range of the scale presented. What is significant is the location of data is below the location of the joint CDF suggesting mechanistic coupling.

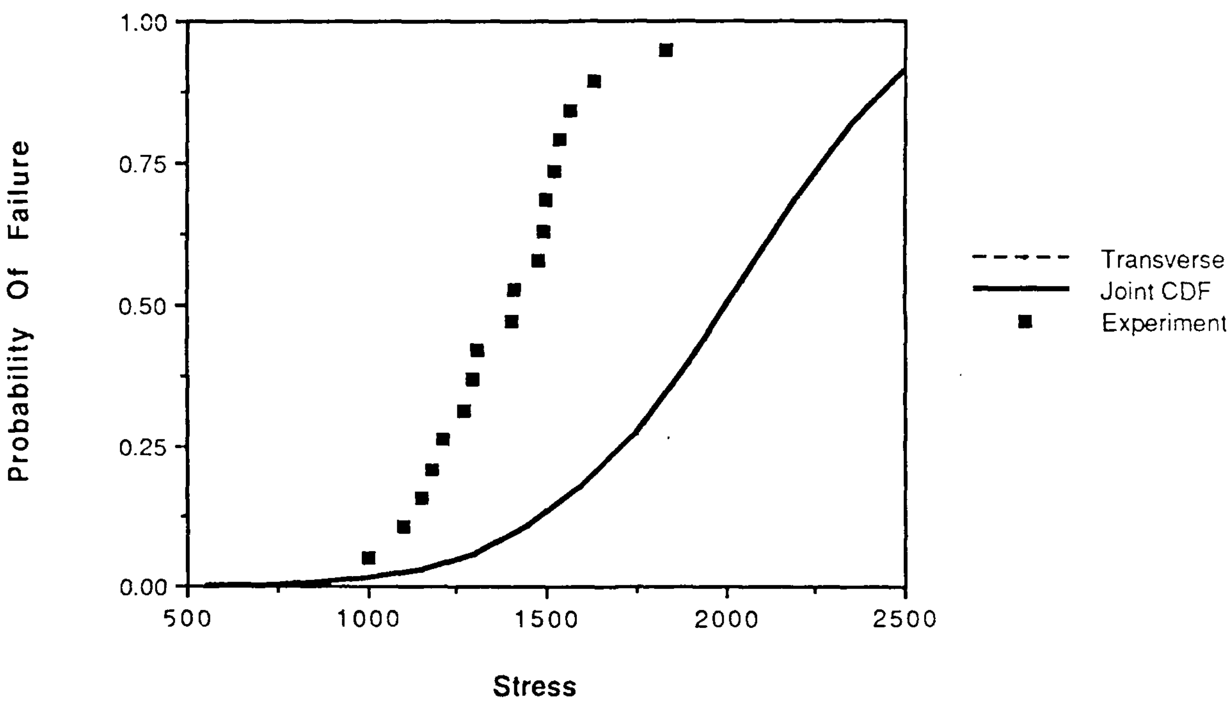

Figure B-5a : Comparison Of Experimental Data With Joint Failure CDF For Independent Case (Linearized, 45 Deg.)

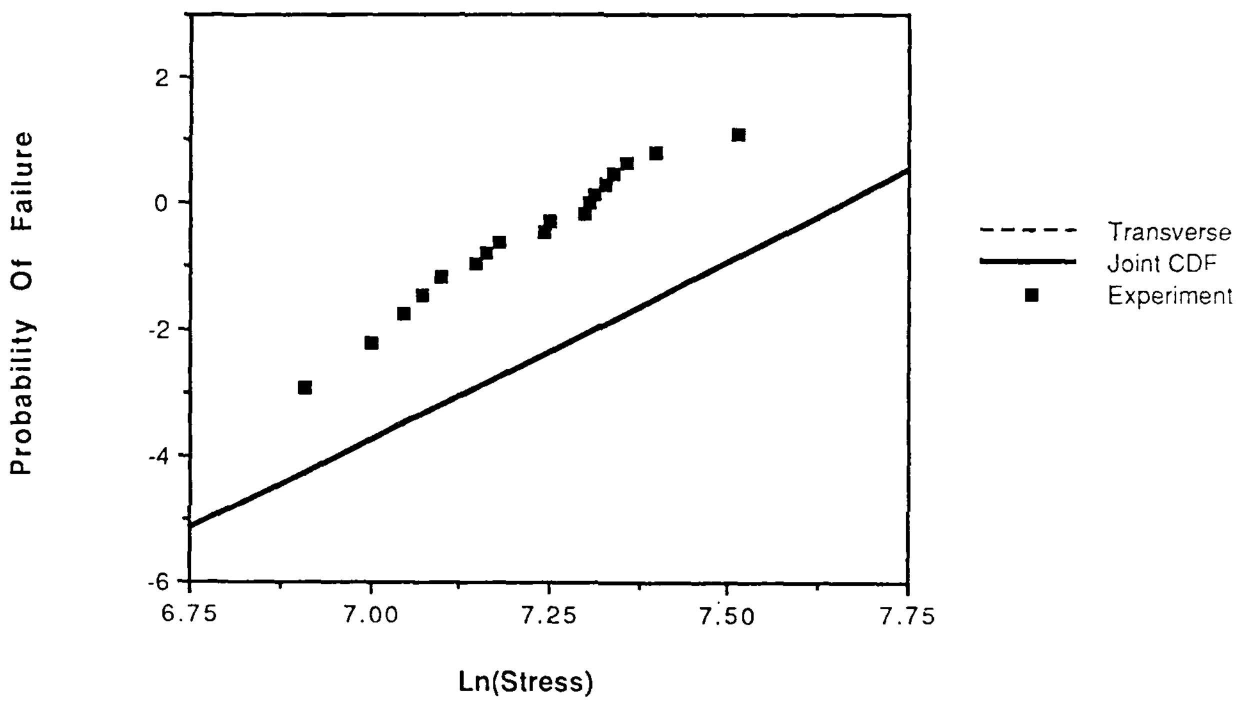

Figure B-5b : Comparison Of Experimental Data With Joint Failure CDF For Independent Case (Linearized, 45 Deg.)

At θ = 60°,Figures B-5a and 5b show that the effect of shear is no longer present and that the fit of the data is much improved. The latter observation further substantiates the existence of mechanistic coupling.

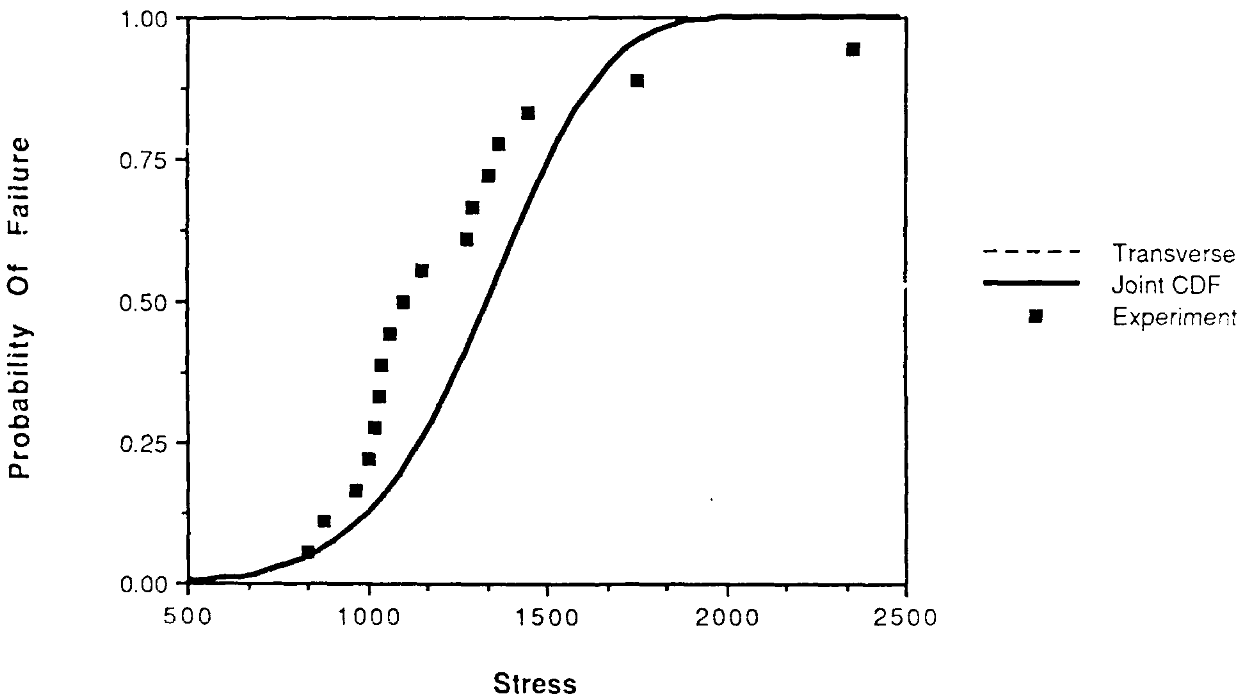

Figure B-6a : Comparison Of Experimental Data With Joint Failure CDF For Independent Case (60 Deg.)

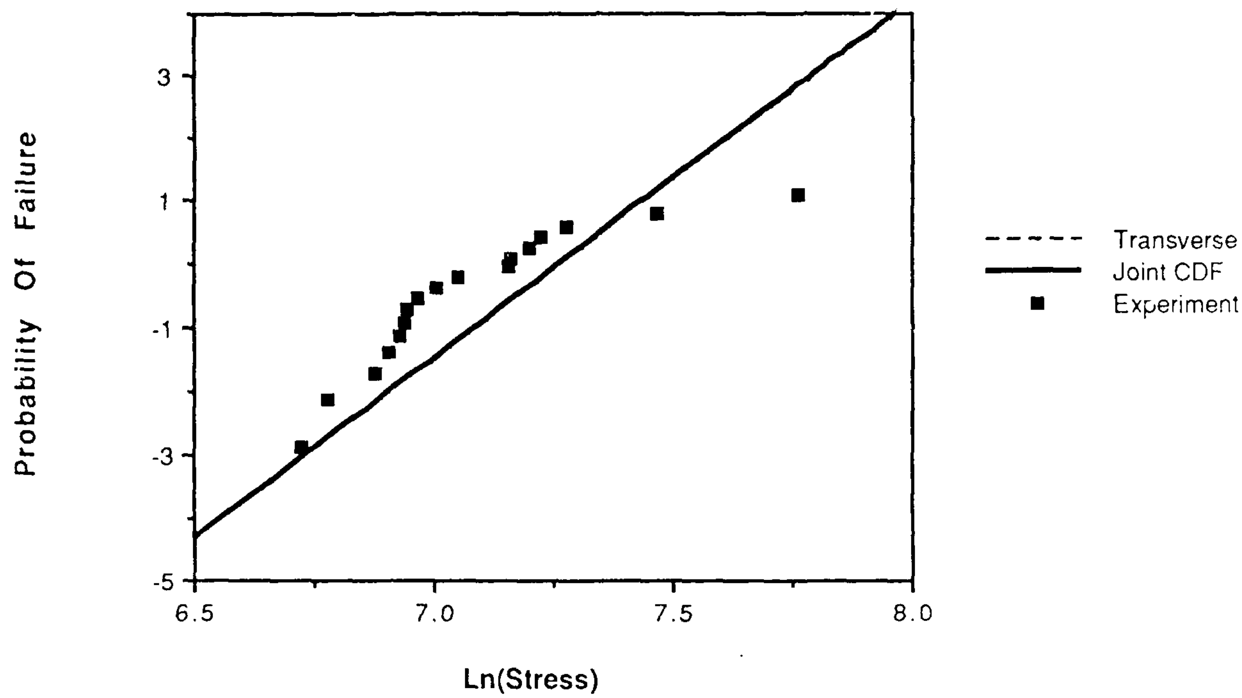

Figure B-6b : Comparison Of Experimental Data With Joint Failure CDF For Independent Case (Linearized, 60 Deg.)

From the experimental data available in the open literature, it was observed that a much larger number of samples are required to identify the probabilistic coupling and dependency of strength. What is evident is that mechanistic coupling needs to be included as an extension to the investigation herein. Furthermore, experimental design using the probability failure criterion is mandatory to optimize experimental testing.

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